

$f(R)$ Theories of Supergravities and Pseudo-supergravitiesHaishan Liu¹, H. Lü^{2,3} and Zhao-Long Wang⁴¹*Zhejiang Institute of Modern Physics**Department of Physics, Zhejiang University, Hangzhou 310027, China*²*China Economics and Management Academy**Central University of Finance and Economics, Beijing 100081, China*³*Institute for Advanced Study, Shenzhen University**Nanhai Ave 3688, Shenzhen 518060, China*⁴*School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea***ABSTRACT**

We present $f(R)$ theories of ten-dimensional supergravities, including the fermionic sector up to the quadratic order in fermion fields. They are obtained by performing the conformal scaling on the usual supergravities to the $f(R)$ frame in which the dilaton becomes an auxiliary field and can be integrated out. The $f(R)$ frame coincides with that of M-theory, D2-branes or NS-NS 5-branes. We study various BPS p -brane solutions and their near-horizon AdS \times sphere geometries in the context of the $f(R)$ theories. We find that new solutions emerge with global structures that do not exist in the corresponding solutions of the original supergravity description. In lower dimensions, We construct the $f(R)$ theory of $\mathcal{N} = 2$, $D = 5$ gauged supergravity with a vector multiplet, and that for the four-dimensional $U(1)^4$ gauged theory with three vector fields set equal. We find that some previously-known BPS singular “superstars” become wormholes in the $f(R)$ theories. We also construct a large class of $f(R)$ (gauged) pseudo-supergravities. In addition we show that the breathing mode in the Kaluza-Klein reduction of Gauss-Bonnet gravity on S^1 is an auxiliary field and can be integrated out.

Contents

1	Introduction	3
2	Converting the gravity/scalar system to $f(R)$	5
2.1	Equations of motion	6
2.2	The conversion	7
2.3	Two examples	9
3	$f(R)$ theories of (gauged) Kaluza-Klein gravities	11
3.1	$f(R)$ Kaluza-Klein gravity	12
3.2	$f(R)$ gauged KK gravity and charged wormholes	14
4	$f(R)$ supergravities in $D = 10$: the bosonic sector	16
4.1	$\mathcal{N} = 1$, $D = 10$ $f(R)$ supergravity	17
4.2	$f(R)$ heterotic supergravity	21
4.3	(Massive) type IIA $f(R)$ supergravity	22
4.4	Type IIB $f(R)$ supergravity	25
4.5	Further discussions on the R-R sector	26
5	$f(R)$ supergravities: the fermionic sector	27
5.1	A general discussion	27
5.2	$\mathcal{N} = 1$, $D = 10$ $f(R)$ supergravity	28
5.3	Type IIA $f(R)$ supergravity	29
6	$f(R)$ gauged (pseudo) supergravities	30
6.1	$\mathcal{N} = 2$, $D = 5$ gauged $f(R)$ supergravity	30
6.2	$D = 4$ gauged supergravity	33
6.3	Gauged $f(R)$ Kaluza-Klein pseudo-supergravities	34
7	A general class of $f(R)$ pseud-supergravities	36
8	Conclusions and discussions	37
A	KK circle reduction without the conformal scaling	40
B	A general class of charged black hole solutions	42

1 Introduction

Under the principle of general coordinate transformation invariance, there are limited ways of generalizing Einstein gravity at the field theoretical level, by adding additional fields and/or including higher-order curvature contributions. Largely motivated by string theory, there have been decades of efforts in constructing supergravities, which are believed to be the low energy effective theories of string or more fundamental M-theory. The study of extended gravities with higher-order curvature terms predated supergravity and string theories. The primary motivation was to render the theory renormalizable by adding higher-order propagators. It turns out that although gravity with higher-order curvature terms can indeed be renormalizable, it suffers from having ghost degrees of freedom [1, 2]. Recently there has been progress in eliminating those ghosts at the classical level for some special regions of the parameter spaces [3, 4, 5, 6]; however, it is unlikely that these parameter regions can survive the renormalization group flow.

If one is to consider only classical or semi-classical generalizations of Einstein gravity, the simplest example is perhaps the $f(R)$ theory in which the Ricci scalar in the Einstein-Hilbert action is replaced by certain appropriate function f of R . Interestingly, there has been almost no overlap in the research areas of supergravity and $f(R)$ gravity. The work of $f(R)$ theories has been primarily focused on cosmology. (See reviews [7, 8, 9].) Recently, we have demonstrated that there exist a subclass of $f(R)$ theories that can admit Killing spinor equations, which allow one to construct “BPS” solutions [10]. Whilst Killing spinor equations can exist in some intrinsically non-supersymmetric theories [11, 12, 13], their appearance is certainly an important characteristic of supergravities. It was shown in [14, 15] that when a bosonic gravity theory admits Killing spinor equations, it can be promoted to pseudo-supergravity which is invariant under the pseudo-supersymmetric transformation rules up to the quadratic order in fermion fields. This suggests that there should exist $f(R)$ pseudo-supergravities. Furthermore, if we add appropriate additional fields so that the degrees of freedom of the bosons and fermions match, we may expect to obtain an $f(R)$ theory of supergravity.

It is well known that $f(R)$ gravity is equivalent to a special class of the Brans-Dicke theory in which the scalar field has no kinetic term. Conversely, any gravity theory coupled to a scalar can be cast into the “ $f(R)$ frame” in which the scalar has no kinetic term and may become auxiliary. Integrating out this auxiliary scalar gives rise to the $f(R)$ theory. Thus, the relation between the $f(R)$ theory and the corresponding Brans-Dicke theory is analogous to that between the Nambu-Goto and Polyakov actions in the string theory.

In this paper, we study supergravities and perform the conformal transformation so that the theories are in the $f(R)$ frame. In general the resulting equation of motion of the scalar field is a polynomial of the scalar with irrational power. However, we find that for all supergravities in $D = 10$, the polynomial is of integer power and hence the scalar can be straightforwardly integrated out, giving rise to the $f(R)$ theory description of these supergravities. We find that this can also be done for $\mathcal{N} = 2$, $D = 5$ gauged supergravity with a vector multiplet, and $D = 4$ $U(1)^4$ gauged theory with three $U(1)$ vectors set to equal. It should be emphasized that although the $f(R)$ theory is equivalent to the Brans-Dicke description at the classical level, they are inequivalent to the original supergravities even at the classical level, since the conformal scaling can be singular in the solution space. We obtain new solutions that are well behaved in the $f(R)$ theories but would be discarded in the original supergravities owing to the bad properties.

The paper is organized as follows. In section 2, we review $f(R)$ gravity and its connection to the special class of Brans-Dicke theory where the scalar has no kinetic term. We then study the conversion of the gravity/scalar system to the $f(R)$ theory. This can be done by first performing a conformal scaling on the theory from the Einstein frame to the $f(R)$ frame and then integrating out the scalar. We present two examples of scalar potentials that appear frequently in supergravities and obtain their corresponding $f(R)$ theories. In particular, we demonstrate that AdS worm-branes, domain walls that connect two AdS Minkowski boundaries, can emerge in the $f(R)$ theories.

In section 3, we study the nature of the $f(R)$ frame. It turns out that if the D -dimensional theory comes from the Kaluza-Klein S^1 reduction of certain $(D+1)$ -dimensional theory, the $f(R)$ frame is in fact the frame of the $(D+1)$ -dimensional metric without any conformal scaling. We then consider a general Lagrangian in arbitrary dimensions involving two vectors and a scalar with a non-trivial scalar potential. This Lagrangian reduces to special cases of the $U(1)^3$ and $U(1)^4$ theories in the $D = 5$ and $D = 4$ gauged supergravities respectively. We construct the $f(R)$ theory of this system, and demonstrate that some singular “black holes” of the original theory become smooth wormholes in the $f(R)$ theory.

In section 4, we construct $f(R)$ theories of ten-dimensional supergravities, focusing on the bosonic sector. We first cast supergravities in the $f(R)$ frame in which the dilaton, which measure the string coupling, becomes an auxiliary field. Integrating out this scalar leads to the $f(R)$ theories of supergravities. For those ten-dimensional supergravities that comes from M-theory on S^1 or S^1/\mathbb{Z}_2 reductions, the $f(R)$ frame is nothing but the M-theory frame, which also coincides with the D2-brane frame and the NS-NS 5-brane frame.

We examine the previously-known p -branes in the context of $f(R)$ theories and show that some previously singular solutions are much better behaved in the $f(R)$ description. For example the usual singular NS-NS string now interpolates between the $\text{AdS}_3 \times S^7$ horizon to the asymptotic flat spacetime. We also obtain new class of p -branes which are supported by delta-function source located at the equator of the foliating sphere in the transverse space. We show that some coordinate of such a solution in the $f(R)$ description has extended range of that in the corresponding local solution in M-theory. This suggests that new non-perturbative physical degrees of freedom can be uncovered by the $f(R)$ theories.

In section 5, we study the fermionic sector of $f(R)$ supergravities. We show that the dilaton remains an auxiliary field even when the fermion sector is included. For $\mathcal{N} = 1$, $D = 10$ and type IIA supergravities, we give the fermionic Lagrangian and the supersymmetric transformation rules up to the quadratic order in fermion fields. We also give the general structure of the $f(R)$ theory involving the gravitino and dilatino fields in general dimensions.

In section 6, we consider $\mathcal{N} = 2$, $D = 5$ gauged supergravity with a vector multiplet. We construct the corresponding $f(R)$ gauged supergravity. We obtain AdS worm-branes and charged wormholes in the $f(R)$ gauged supergravity. We extend the discussion to $D = 4$ gauged supergravity and also the gauged $f(R)$ Kaluza-Klein pseudo-supergravity. In section 7, we construct a large class of $f(R)$ pseudo-supergravities in general dimensions. We give the conclusions and present further discussions in section 8. In appendix A, we study the Kaluza-Klein circle reduction where the lower-dimensional metric is not scaled by the breathing mode. We demonstrate that the breathing mode is auxiliary even when the higher-order Gauss-Bonnet curvature term is included. In appendix B, we present a general class of charged black hole solutions of the theory discussed in section 3.2.

2 Converting the gravity/scalar system to $f(R)$

In this section, we shall give a quick review of $f(R)$ gravity, which is defined by replacing the Ricci scalar R in the Einstein-Hilbert action with an appropriate function f of R . For non-vanishing $f''(R)$, the theory was known to be related to a special class of the Brans-Dicke theory. We shall study this relation focusing on the conversion of a gravity/scalar system to $f(R)$ gravity. We consider two explicit examples of scalar potentials that arise in supergravities. We demonstrate that the two systems are not equivalent, owing to the possibility that singular conformal scaling may arise in the solution space. In other words, the $f(R)$ theories can have solutions with global properties that do not exist in the corresponding

local solutions of the gravity/scalar systems of supergravities.

2.1 Equations of motion

The Lagrangian of $f(R)$ gravity in general dimensions is given by

$$e^{-1}\mathcal{L}_D = f(R), \quad (1)$$

where $e = \sqrt{-\det(g_{\mu\nu})}$. (There should be no confusion between this e which appears only as e^{-1} in this paper and the notation for the exponential function.) In this paper, we shall be concerned with only $f(R)$ theories in the metric formalism, and hence the equations of motion from the variation of $g_{\mu\nu}$ are given by

$$\mathcal{G}_{\mu\nu} \equiv F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)F(R) = 0, \quad (2)$$

where $F(R) = f'(R)$. Note that in this paper, we always use a prime to denote a derivative with respect to R , unless an explicit new variable is given. Taking the trace, we have

$$\mathcal{R} \equiv RF - \frac{1}{2}Df + (D-1)\square F = 0. \quad (3)$$

The equations of motion (2) can be equivalently expressed as

$$\mathcal{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{F}\nabla_\mu\nabla_\nu F + \frac{1}{2(D-1)F}(f - 2RF)g_{\mu\nu} = 0. \quad (4)$$

The simplest class of solutions for $f(R)$ gravities are perhaps the Einstein metrics, for which R is a constant, which we denote as $R_0 \equiv D\Lambda$, where Λ is the effective cosmological constant. It follows from (3) that

$$2R_0F(R_0) = Df(R_0). \quad (5)$$

Depending on whether Λ is positive, 0, or negative, the vacuum solution is de Sitter (dS), Minkowski and anti-de Sitter (AdS) respectively. In the special case when $f(R_0) = 0 = F(R_0)$, the full equations of motion are reduced to simply the scalar type of equation $R = R_0$, and the theory has no propagating spin-2 degrees of freedom [10]. It was shown in [10] that there exists a subclass of $f(R)$ theories that admit Killing spinor equations. Exact non-trivial ‘‘BPS’’ domain walls and cosmological solutions with varying R are consequently obtained. It was demonstrated that $f(R)$ theories can admit Einstein metrics that do not satisfy (3); they are characterized by the divergent $F(R_0)$ [10].

2.2 The conversion

It is well-known that $f(R)$ gravity can be cast into the form of the Brans-Dicke theory. To see this, one starts with the Lagrangian

$$e^{-1}\mathcal{L} = f(\chi) + f_{,\chi}(\chi)(R - \chi). \quad (6)$$

Variation with respect to χ gives rise to

$$f_{,\chi\chi}(R - \chi) = 0. \quad (7)$$

Thus provided that $f_{,\chi\chi} \neq 0$, one has $\chi = R$, and hence (6) gives rise to the usual $f(R)$ theory. On the other hand, we can treat χ as a scalar field and hence the Lagrangian (6) is a gravity/scalar system. To make this manifest, one can define

$$\varphi = f_{,\chi}(\chi), \quad (8)$$

and hence the $f(R)$ gravity is equivalent to the Brans-Dicke theory of the type

$$e^{-1}\mathcal{L} = \varphi R + f(\chi(\varphi)) - \varphi\chi(\varphi). \quad (9)$$

This is a special class of Brans-Dicke theory with no manifest kinetic term for φ . The conversion of $f(R)$ gravity to the Brans-Dicke theory requires finding the inverse function of $F = f'$, which in general does not have explicit analytical form. The absence of the kinetic term for φ implies that the variation of φ gives rise to a pure algebraic equation for φ .

Conversely, one may convert some gravity/scalar systems to $f(R)$ theories. In supergravities, the Lagrangians are typically written in the Einstein frame. For now we consider gravity coupled to a single scalar only:

$$e^{-1}\mathcal{L}_D = R - \frac{1}{2}(\partial\phi)^2 - V(\phi). \quad (10)$$

Let us make the following conformal transformation and the field redefinition for the dilaton

$$g_{\mu\nu} \rightarrow e^{-2\kappa\alpha\phi} g_{\mu\nu}, \quad \varphi = e^{\kappa\beta\phi}, \quad (11)$$

with $\kappa = \pm 1$ and

$$\alpha = -\frac{1}{\sqrt{2(D-1)(D-2)}}, \quad \beta = \sqrt{\frac{D-2}{2(D-1)}}. \quad (12)$$

It is clear that this conformal transformation can be done in all dimensions greater or equal to three. Note that these two constants are rational numbers only in $D = 3$ and $D = 10$. The Lagrangian (10) becomes

$$e^{-1}\mathcal{L}_D = \varphi R - \tilde{V}(\varphi), \quad (13)$$

where

$$\tilde{V}(\varphi) = \varphi^{\frac{D}{D-2}} V(\phi(\varphi)), \quad (14)$$

Comparing to (9), we find that if the theory can be converted into an $f(R)$ theory, f must satisfy the following

$$f(R) - RF(R) + \tilde{V}(F) = 0. \quad (15)$$

For the situation with non-vanishing $f''(R)$, acting with ∂_R , we have

$$\frac{d\tilde{V}}{dF} = R, \quad (16)$$

This is a purely algebraic equation for the function $F(R)$. Once we solve for F , the $f(R)$ theories can be derived by a first-order integration.

A more direct approach is to view the φ in (13) as an auxiliary field since it does not have a kinetic term. The equation of motion associated with the variation of φ is $d\tilde{V}/d\varphi = R$, which is an algebraic equation for φ . Solving for φ and substituting it back into (13), we obtain the corresponding $f(R)$ theory. Thus the relation between the $f(R)$ theory and the Brans-Dicke theory (13) is analogous to that between the Nambu-Goto and Polyakov actions of the string theory, and hence they are classically equivalent. Of course, it is not always possible to get a close form solution even for an algebraic equation. However, it should be emphasized that the $f(R)$ theory can be inequivalent to the original gravity/scalar system (10) since the conformal scaling can be singular. Note that $\varphi = F(R)$ and hence a necessary ghost-free condition is that $F(R)$ is non-negative.

A special case should be addressed in which the potential function \tilde{V} is linear in φ , *i.e.*

$$\tilde{V} = a\varphi + b, \quad \longrightarrow \quad e^{-1}\mathcal{L}_D = \varphi(R - a) - b. \quad (17)$$

Substituting \tilde{V} in (15), we have an $f(R)$ that is linear on R , *i.e.* the usual Einstein gravity with a cosmological constant. The converted theory is inequivalent to the original theory, owing to the fact $f''(R) = 0$ in this case. In fact the equation of motion for φ gives no information on φ , but simply tells us that $R = a$. Thus the gravity/scalar system cannot be converted into $f(R)$ theory with φ absorbed as F . However, in many explicit examples,

the Lagrangian (17) can be viewed as a limiting one of a more general $f(R)$ theory. The $f(R)$ limit to Einstein gravity with $f''(R) \rightarrow 0$ was studied in [16].

It is worth remarking that in higher-derivative theories, we can expect that \tilde{V} in (13) is a function not only of φ , but also of spacetime derivatives of φ . Then, φ ceases to be auxiliary. Thus it is much more non-trivial for φ to be auxiliary in higher-derivative gravities. Further discussion and an explicit example are provided in appendix A.

As we have mentioned, for a generic function V , there is no analytical solution to the equation (16). We shall give two examples of V that are relevant to this paper, for which analytical solutions for f can be found.

2.3 Two examples

In this subsection, we consider two classes of scalar potentials that frequently appear in supergravities.

Example 1: The first example is just a pure exponential potential

$$V(\phi) = \frac{1}{2}m^2 e^{-D\alpha b\phi}, \quad (18)$$

where α is given in (12) and b is an arbitrary constant parameter. The scalar potential in massive type IIA supergravity [17] is of this form. Some scalar potentials in gauged supergravities also take this form but with negative m^2 . (See *e.g.*, [18, 19, 20, 21].) Furthermore, the dilaton coupling with the form-fields is of this form if we can treat the form-fields as constant. It is straightforward to obtain the corresponding $f(R)$:

$$f(R) = \frac{2 + \kappa D b}{D(1 + \kappa b)} R \left(\frac{2(D-2)R}{D(1 + \kappa b)m^2} \right)^{\frac{D-2}{2+\kappa D b}}. \quad (19)$$

Note that there is an identity between f and F , given by

$$f = \frac{2 + \kappa D b}{D(1 + \kappa b)} F R, \quad (20)$$

This identity allows us to take a smooth limit of sending $m \rightarrow 0$ and recover the Einstein theory with a free scalar. To see this, note that if we let m vanish, the regularity of the $f(R)$ theory (19) requires that R approaches zero. There is no unique way of how these two quantities approach zero, and the ratio can be described by a scalar quantity. If we define $\varphi = F$, and in the $m \rightarrow 0$, it follows from (20) that the Lagrangian becomes

$$e^{-1}\mathcal{L} \sim \varphi R. \quad (21)$$

This Lagrangian is consistent with the limit $R \rightarrow 0$ since the variation of φ gives rise to $R = 0$. Converting this Lagrangian to Einstein frame, we obtain the original Einstein theory with a free scalar.

In general the parameter b in (18) is irrational in supergravities. The resulting $f(R)$ theory will have irrational power of R . We consider this as an unnatural formalism of a gravity theory. We shall focus our attention in examples where b is rational. An interesting case is $b = 0$. The scalar/gravity system is simply the cosmological Einstein gravity with a “massless” free scalar. The resulting $f(R)$ theory becomes

$$f(R) \propto R^{\frac{D}{2}}. \quad (22)$$

This $f(R)$ is interesting in that all Einstein metrics with any effective cosmological constants are solutions.

It should be remarked that if $(D - 2)/(2 + \kappa Db) > 0$, the $f(R)$ theory (19) admits a solution $R = 0$. Such a solution clearly does not exist in the original gravity/scalar theory.

Example 2: The second example is a scalar potential involving two specific exponential terms

$$\begin{aligned} V &= -(D - 1) \left((D - 3)g_1^2 e^{2\alpha\phi} + g_2^2 e^{-2(D-3)\alpha\phi} \right) \\ &= -(D - 1) \left((D - 3)g_1^2 e^{-\sqrt{\frac{2}{(D-1)(D-2)}}\phi} + g_2^2 e^{\frac{\sqrt{2}(D-3)}{\sqrt{(D-1)(D-2)}}\phi} \right). \end{aligned} \quad (23)$$

One reason for us to consider this scalar potential is that as we shall see in the next section, it can be embedded in various gauged supergravities. This scalar potential can be expressed in terms of a superpotential W , namely [13]

$$\begin{aligned} V &= \left(\frac{dW}{d\phi} \right)^2 - \frac{D - 1}{2(D - 2)} W^2, \\ W &= \frac{1}{\sqrt{2}} \left(g_1^2 g_2^{-1} (D - 3) e^{-\frac{D-1}{\sqrt{2(D-1)(D-2)}}\phi} + (D - 1) g_2 e^{\frac{D-3}{\sqrt{2(D-1)(D-2)}}\phi} \right). \end{aligned} \quad (24)$$

Under the conformal transformation (11), the Lagrangian becomes

$$e^{-1}\mathcal{L} = \varphi \left(R + (D - 1)(D - 3)g_1^2 \right) + (D - 1)g_2^2 \varphi^3. \quad (25)$$

Note that if we simply set $g_2 = 0$, the theory cannot be converted to the $f(R)$ formalism. In general, it follows from (16), we have

$$F = \frac{\sqrt{-R - (D - 1)(D - 3)g_1^2}}{\sqrt{3(D - 1)g_2^2}}. \quad (26)$$

The corresponding $f(R)$ is given by

$$f = \frac{2}{3}F(R) \left(R + (D-1)(D-3)g_1^2 \right). \quad (27)$$

Thus we have obtained the $f(R)$ theory from the gravity/scalar system (23). In the limit of $g_2 \rightarrow 0$, we reproduce the Lagrangian (25) up to an overall scaling factor $2/3$.

Although the $f(R)$ theory (27) with (26) is obtained from the gravity/scalar system (10) with the potential (23), the two theories should not be considered as equivalent, even at the classical level. It follows from (26) and (27) that we have $f(R_0) = 0 = F(R_0)$, where $R_0 = -(D-1)(D-3)g_1^2$. Thus any metrics with $R = R_0$ is a solution. Such a solution does not exist in the original gravity/scalar system, demonstrating that the two theories are not equivalent.

As we shall discuss in section 6, it turns out that the $f(R)$ theory (27) admits the following Killing spinor equations

$$D_\mu \epsilon + \frac{1}{2}gF\Gamma_\mu \epsilon = 0, \quad \Gamma^\mu \partial_\mu F \epsilon - \frac{1}{2}(D-3)(F^2 - 1)\epsilon = 0, \quad (28)$$

where we have set $g_1 = g = g_2$. Following the technique of [10], we find the following ‘‘BPS’’ domain-wall solution

$$ds^2 = dr^2 + \left(\cosh\left(\frac{1}{2}(D-3)g r\right) \right)^{\frac{4}{D-3}} dx^\mu dx_\mu, \quad F = \tanh\left(\frac{1}{2}(D-3)g |r|\right). \quad (29)$$

This solution describes an AdS worm-brane, connecting two AdS Minkowski boundaries at $r = \pm\infty$ without a horizon in between. Note the absolute-value sign on r is imposed in F to ensure the solution is ghost free. It follows that a matter delta-function source at $r = 0$ is necessary to sustain this worm-brane. In the original gravity/scalar system, this AdS worm-brane would become a singular domain wall with a power-law curvature singularity at $r = 0$. Thus, although the local solutions are related by the conformal transformation, the globally-defined worm-brane with r running from $-\infty$ to $+\infty$ does not exist in the original gravity/scalar theory.

3 $f(R)$ theories of (gauged) Kaluza-Klein gravities

In the previous section, we present the formalism and examples of converting a system of gravity coupled with a single scalar to $f(R)$ theories. The necessary conformal scaling (11) is universal, independent of the scalar potential, and the constants (α, β) are remarkably the same as those associated with the scaling factors in the Kuluza-Klein circle reduction [22]. It

is thus of interest to investigate the connection between the $f(R)$ frame and the dimensional reduction. Also we would like to investigate whether we can convert some supergravity theories involving form fields into the $f(R)$ formalism. We shall study both questions by examining the Kaluza-Klein theory that is S^1 reduction of pure Einstein gravity. Starting from Einstein gravity in $(D+1)$ dimensions

$$\hat{e}^{-1}\mathcal{L}_{D+1} = \hat{R}, \quad (30)$$

we perform the circle reduction with the metric ansatz [22]

$$d\hat{s}_{D+1}^2 = e^{2\alpha\phi}ds_{\text{Ein}}^2 + e^{2\beta\phi}(dz + \mathcal{A}_{(1)})^2, \quad (31)$$

where α and β are given by (12). The metric ds_{Ein}^2 in lower dimensions is in the Einstein frame. The resulting D -dimensional Lagrangian is

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}_{(2)}^2. \quad (32)$$

where $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$. What we like to draw attention here is that the conformal transformation (11) implies that

$$d\hat{s}_{D+1}^2 = e^{2(1-\kappa)\alpha\phi}ds_D^2 + e^{2\beta\phi}(dz + \mathcal{A}_{(1)})^2. \quad (33)$$

For $\kappa = +1$, we have

$$d\hat{s}_{D+1}^2 = d\hat{s}_D^2 + \varphi^2(dz + \mathcal{A}_{(1)})^2. \quad (34)$$

Thus the metric in D dimensions is in the same frame as that in $(D+1)$ dimensions, without any conformal scaling. For $\kappa = -1$, The reduction ansatz becomes

$$ds_{D+1}^2 = \varphi^{\frac{4}{D-2}}ds_D^2 + \varphi^{-2}(dz + \mathcal{A}_{(1)})^2, \quad (35)$$

This does not appear to have a particular interesting physical interpretation, and hence we shall focus our attention primarily on $\kappa = +1$. Thus if a D -dimensional theory has an origin in $(D+1)$ dimensions, we may define the “ $f(R)$ frame” simply as the $(D+1)$ -dimensional frame.

3.1 $f(R)$ Kaluza-Klein gravity

With the reduction ansatz (34), the Lagrangian of Kaluza-Klein gravity in D dimensions in the $f(R)$ -frame is

$$e^{-1}\mathcal{L}_D = \varphi R - \frac{1}{4}\varphi^3\mathcal{F}_{(2)}^2. \quad (36)$$

Thus we see that the breathing mode φ is an auxiliary scalar field. In appendix A, we present the Kaluza-Klein reduction with the metric ansatz (34). We show that when a generic higher-order curvature term is included in $(D + 1)$ dimensions, the scalar φ ceases to be auxiliary. However, it remains an auxiliary field for the circle reduction of Gauss-Bonnet gravity.

In the procedure of converting the Kaluza-Klein theory (36) to an $f(R)$ theory, we can treat $\mathcal{F}_{(2)}^2$ as if it is a constant. The Lagrangian is then analogous to the first example discussed in section 2.3. It follows from the discussion there that $F(R) = 2\sqrt{R/(3\mathcal{F}_{(2)}^2)}$. Thus the $f(R)$ theory of the Kaluza-Klein gravity (32) is given by

$$e^{-1}\mathcal{L}_D = f(R) = \frac{4}{3}\sqrt{\frac{R^3}{3\mathcal{F}_{(2)}^2}}. \quad (37)$$

To demonstrate that this derivation with $\mathcal{F}_{(2)}^2$ being treated as a constant is legitimate, we give the two equations of motion associated with $\delta\mathcal{A}_\mu$ and $\delta g^{\mu\nu}$:

$$\begin{aligned} \nabla_\mu(F^3\mathcal{F}^{\mu\nu}) &= 0, \\ FR_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)F - \frac{1}{2}F^3(\mathcal{F}_{(2)}^2)_{\mu\nu} &= 0. \end{aligned} \quad (38)$$

It is easy to verify that these equations of motion are the same as the ones derive from (36), and hence they are also equivalent to those from (32), up to the conformal scaling which may be singular. Note that in this paper, we use notation f and F exclusively for $f(R)$ and $F(R)$. The notation F should not be confused with form fields which either carry explicit indices or the subscript indicating the rank of the form.

It is worth remarking that to be pedantic the proper f expression should be $f = 4/3R\sqrt{R/(3\mathcal{F}_{(2)}^2)}$. Since $\sqrt{R^2} = \pm R$ depending on the sign of R , there may be an overall minus sign in the Lagrangian (37). In this paper, we shall not be always precise regarding this overall sign of the Lagrangian, unless the issue of the ghost-free condition is discussed.

If the $(D+1)$ -dimensional Einstein gravity is coupled to a cosmological constant, namely

$$\hat{e}^{-1}\mathcal{L}_{D+1} = \hat{R} - \Lambda_0, \quad (39)$$

The reduced theory becomes

$$e^{-1}\mathcal{L}_D = \varphi(R - \Lambda_0) - \frac{1}{4}\varphi^3\mathcal{F}_{(2)}^2. \quad (40)$$

The corresponding $f(R)$ is given by

$$f(R) = \frac{4}{3}\sqrt{-\frac{(\Lambda_0 - R)^3}{\mathcal{F}_{(2)}^2}}. \quad (41)$$

The form of the equations of motion is identical to (38).

A natural question to ask is what happens when $\mathcal{F}_{(2)}$ vanishes. In this case, the regularity of the action requires that $R = \Lambda_0$. It is clear that there is no unique way how the two quantities $\mathcal{F}_{(2)}$ and $(R - \Lambda_0)$ vanish. There should a scalar field describing the ratio of these two quantities when they approaches zero. Considering the identity

$$f(R) = \frac{2}{3}F(R)(R - \Lambda_0), \quad (42)$$

for any $\mathcal{F}_{(2)}$, it is clear that $F = \varphi$ is the scalar field, and hence we recover (40) with $\mathcal{F}_{(2)} = 0$. In this case, the constraint $R = \Lambda_0$ is the equation of motion for the φ field.

Finally, let us consider $(D + 1)$ -dimensional Einstein gravity coupled to an n -form field strength

$$\hat{e}^{-1}\mathcal{L}_{D+1} = \hat{R} - \frac{1}{2n!}\hat{F}_{(n)}^2. \quad (43)$$

It is straightforward to see that the Kaluza-Klein theory in the $f(R)$ frame is given by

$$\begin{aligned} e^{-1}\mathcal{L} &= \varphi(R - \frac{1}{2n!}F_{(n)}^2) - \frac{1}{2(n-1)!}\varphi^{-1}F_{(n-1)}^2 - \frac{1}{4}\varphi^3\mathcal{F}_{(2)}^2, \\ F_{(n)} &= dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)}, \quad F_{(n-1)} = dA_{(n-2)}, \quad \mathcal{F}_{(2)} = d\mathcal{A}_{(1)}. \end{aligned} \quad (44)$$

Applying this result to M-theory, it can be easily deduced that the $f(R)$ frame for ten-dimensional supergravities is the same as that of the M-theory, D2-branes or NS-NS 5-branes.

3.2 $f(R)$ gauged KK gravity and charged wormholes

It was demonstrated that the Kaluza-Klein theory in any dimensions can be pseudo-supersymmetrized by the inclusion of a pseudo-gravitino and dilatino. The full Lagrangian is invariant under the pseudo-supersymmetric transformation rules up to the quadratic fermion order [15]. Furthermore, the pseudo-supergravity can be gauged and the fermions are all charged under the Kaluza-Klein vector. The gauging generates a scalar potential (23). The full bosonic Lagrangian is given by [13, 15]

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}_{(2)}^2 - (D-1)g^2\left((D-3)e^{2\alpha\phi} + e^{-2(D-3)\alpha\phi}\right), \quad (45)$$

where α is given in (12). Note that the scalar potential was discussed as the second example in section 2.3. This theory can be embedded in gauged supergravities in $D = 4, 5$ and 7 [13]. (See also, for example, [23].) In the case of $D = 6$, it may also be possible to embed the theory in the $F(4)$ gauged supergravity [24] coupled to a vector multiplet [25, 26]. Under

the conformal scaling with $\kappa = +1$, we have

$$e^{-1}\mathcal{L} = \varphi\left(R + (D-1)(D-3)g^2\right) + \varphi^3\left(-\frac{1}{4}\mathcal{F}_{(2)}^2 + g^2(D-1)\right), \quad (46)$$

The corresponding $f(R)$ is given by

$$f(R) = \frac{4}{3\sqrt{3}}\sqrt{\frac{\left(-R - (D-1)(D-3)g^2\right)^3}{4(D-1)g^2 - \mathcal{F}_{(2)}^2}}. \quad (47)$$

Again, the form of the equations of motion is identical to (38).

In fact the Lagrangian (45) can be generalized to include another vector and a tensor as well, giving

$$\begin{aligned} e^{-1}\mathcal{L} = & R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-4\alpha\phi}F_{(3)}^2 - \frac{1}{4}e^{2(D-3)\alpha\phi}F_{(2)}^2 - \frac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}_{(2)}^2 \\ & + (D-1)g^2\left((D-3)e^{-\sqrt{\frac{2}{(D-1)(D-2)}}\phi} + e^{\frac{\sqrt{2}(D-3)}{\sqrt{(D-1)(D-2)}}\phi}\right), \end{aligned} \quad (48)$$

where $F_{(3)} = dA_{(2)} - \mathcal{A}_{(1)} \wedge dA_{(1)}$, $F_{(2)} = dA_{(1)}$ and $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$. If we set $g = 0$, the theory is the S^1 reduction of Einstein gravity coupled to a 3-form field strength in $(D+1)$ dimensions, whose corresponding $f(R)$ theory was given in (44) with $n = 3$. One reason that we are interested in this theory is that it admits multi-charge AdS black hole solutions, as we shall demonstrate in appendix B. Furthermore, as we shall discuss in section 6, this Lagrangian can also be embedded in both $D = 4$ and $D = 5$ gauged supergravities. In the $f(R)$ frame, the theory is given by

$$e^{-1}\mathcal{L} = \varphi\left(R + (D-1)(D-3)g^2 - \frac{1}{12}F_{(3)}^2\right) + \varphi^3\left(-\frac{1}{4}\mathcal{F}_{(2)}^2 + g^2(D-1)\right) - \frac{1}{4}\varphi^{-1}F_{(2)}^2. \quad (49)$$

The variation of φ gives rise to a quadratic equation of motion for φ^2 , and hence the $f(R)$ theory can be obtained straightforwardly. The expression is clumsy and we shall not present it here. If we set $F_{(2)} = 0$, the $f(R)$ takes the similar form as (47) with the $F_{(3)}^2$ term appropriately inserted. If instead $\mathcal{F}_{(2)}$ is such that $\mathcal{F}_{(2)}^2 = 4g^2(D-1)$, we have

$$\begin{aligned} f(R) &= F\left(R + (D-1)(D-3)g^2 - \frac{1}{12}F_{(3)}^2\right), \\ F &= \sqrt{-\frac{F_{(2)}^2}{4\left(R + (D-1)(D-2)g^2 - \frac{1}{12}F_{(3)}^2\right)}}. \end{aligned} \quad (50)$$

As we have mentioned, in appendix B, we shall give a general class of non-extremal static multi-charged black hole solutions for the system (49). The non-extremal parameter can be turned off while keeping the charge parameters fixed. The resulting solution is given by

$$ds_D^2 = -\mathcal{H}^{-\frac{D-3}{D-2}}H^{-\frac{D-1}{D-2}}h\,dt^2 + \mathcal{H}^{\frac{1}{D-2}}H^{\frac{D-1}{(D-2)(D-3)}}\left(\frac{dr^2}{h} + r^2d\Omega_{D-2}^2\right),$$

$$\begin{aligned}
\mathcal{F}_{(2)} &= dt \wedge d\mathcal{H}^{-1}, & F_{(2)} &= \sqrt{\frac{D-1}{D-3}} dt \wedge dH^{-1}, \\
h &= 1 + g^2 r^2 \mathcal{H} H^{\frac{D-1}{D-3}}, & e^\phi &= \left(\frac{\mathcal{H}}{H}\right)^{\sqrt{\frac{D-1}{2(D-2)}}}, \\
\mathcal{H} &= 1 + \frac{\tilde{q}}{r^{D-3}}, & H &= 1 + \frac{q}{r^{D-3}}.
\end{aligned} \tag{51}$$

In the $f(R)$ frame, the metric becomes much simpler:

$$ds_D^2 = -(\mathcal{H}H)^{-1} h dt^2 + H^{\frac{2}{D-3}} (h^{-1} dr^2 + r^2 d\Omega_{D-2}^2). \tag{52}$$

The solution has a naked curvature power-law singularity in both frames. This is because $r = 0$ is not a horizon, but instead the metric has the form $ds^2 \sim r^{2(D-4)} dr^2 - dt^2 + d\Omega_{D-2}^2$, and hence the natural radial coordinate is $\rho = r^{D-3}$. When ρ becomes negative such that either H or \mathcal{H} vanishes, the metric has a power-law curvature singularity. As we shall discussed in section 6, in $D = 4, 5$, the theory is part of gauged supergravities and these solutions are BPS and called “superstars.”

Now if instead we set $\tilde{q} = 0$, the metric in the $f(R)$ frame (52) near $r = 0$ behaves like $ds^2 \sim r^{D-5} dr^2 - dt^2 + d\Omega_{D-2}^2$ and thus the natural coordinate should be $\rho = r^{(D-3)/2}$. In terms of the ρ coordinate the solution is given by

$$\begin{aligned}
ds_D^2 &= -\frac{h}{\rho^2 + q^2} dt^2 + (\rho^2 + q^2)^{\frac{2}{D-3}} \left(\frac{4d\rho^2}{(D-3)^2 h} + d\Omega_{D-2}^2 \right), \\
A_{(1)} &= \sqrt{\frac{D-1}{D-3}} \frac{q}{\rho^2 + q} dt, & h &= \rho^2 + g^2(\rho^2 + q)^{\frac{D-1}{D-3}}, & \varphi \equiv F &= \frac{|\rho|}{\rho^2 + q^2},
\end{aligned} \tag{53}$$

It is clear that this solution describes a wormhole with the radial coordinate ρ running from $-\infty$ to $+\infty$, connecting two $\mathbb{R}^t \times S^{D-2}$ boundaries. The positivity condition for $\varphi \equiv F$ require that an absolute-value sign be added on ρ in its expression, and hence the Einstein equations of motion (2) requires that a delta-function matter source be needed to support this wormhole. Note that the level surfaces for this wormhole are $\mathbb{R}^t \times S^{D-2}$, unlike the worm-brane discussed earlier. If we convert this solution to that of the original (48) theory, then it has a power-law curvature singularity at $\rho = 0$. Thus the charged wormhole of our $f(R)$ theory with $r \in (-\infty, +\infty)$ does not exist in the original theory (48).

4 $f(R)$ supergravities in $D = 10$: the bosonic sector

Having addressed the preliminaries in sections 2 and 3, we now turn our attention to converting supergravities to the $f(R)$ formalism. Eleven-dimensional supergravity [27] can be argued as the most fundamental one; however, since there is no scalar mode in presence,

there can be no $f(R)$ formalism. In this section, we consider $f(R)$ supergravities in ten dimensions. We shall focus our attention only on the bosonic sector. The discussion of fermions will be given in section 5.

4.1 $\mathcal{N} = 1$, $D = 10$ $f(R)$ supergravity

The simplest supergravity in $D = 10$ has $\mathcal{N} = 1$ supersymmetry [28]. The field content for the bosonic sector consists of the metric, a dilaton ϕ and a 2-form antisymmetric tensor field $A_{\mu\nu}$. The bosonic Lagrangian is given by

$$e^{-1}\mathcal{L}_{10} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-\phi}F_{(3)}^2, \quad (54)$$

Making a conformal transformation (11) with $\kappa = +1$, we have the theory in the $f(R)$ frame

$$e^{-1}\mathcal{L}_{10} = \varphi R - \frac{1}{12}\varphi^{-1}F_{(3)}^2, \quad (55)$$

and hence the corresponding $f(R)$ theory is

$$e^{-1}\mathcal{L}_{10} = f(R) = \sqrt{-\frac{1}{3}RF_{(3)}^2}. \quad (56)$$

The equations of motion are given by

$$\begin{aligned} FR_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)F - \frac{1}{4}F^{-1}(F_{(3)}^2)_{\mu\nu} &= 0, \\ \nabla_\mu(F^{-1}F^{\mu\nu\rho}) &= 0. \end{aligned} \quad (57)$$

Note that we have an identity $f = 2FR$. This implies that in the limit of $F_{(3)}^2$ goes to zero, the quantity F should be viewed as a scalar quantity that is held fixed, leading to (55) with the vanishing 3-form. The Hodge dual of $F_{(3)}$ is a 7-form $F_{(7)}$. The dual description of (55) is given by

$$e^{-1}\mathcal{L}_{10} = \varphi(R - \frac{1}{10080}F_{(7)}^2). \quad (58)$$

This implies that the $f(R)$ frame is in fact that of NS-NS 5-branes.

Ten-dimensional $\mathcal{N} = 1$ supergravity can be viewed as part of the lower-energy effective theory of string. The dilaton ϕ plays the role as the string loop expansion field. The string coupling constant is $g_s = \langle e^\phi \rangle$. Since $\varphi = e^{\frac{4}{3}\phi}$, it follows that we have

$$\langle \varphi \rangle = g_s^{4/3}. \quad (59)$$

An important feature in the theory (55) is that the scalar φ is no longer a dynamical field of an independent degree of freedom, but instead it is given algebraically by $\varphi =$

$\sqrt{-F_{(3)}^2/(12R)}$. As we mentioned in section 2, the relation between the $f(R)$ theory (56) and the Brans-Dicke like theory (55) is analogous to that of the Nambu-Goto and Polyakov actions. The description of (56) is of the non-perturbative nature with the scalar associated with the loop expansions absorbed as part of $f(R)$. Whilst the two theories (56) and (55) are classically equivalent, they are not equivalent to the original supergravity (54) since the conformal scaling can be singular.

It should be remarked that it is perhaps a misnomer to continually call (56) as an $f(R)$ theory, since it also contains matter field $F_{(3)} = dA_{(2)}$, which *is not* a constant, but a dynamical field. Nevertheless, we shall continue use the terminology $f(R)$ owing to the lack of any elegant alternative. Note that in our $f(R)$ supergravities, the matter field $F_{(3)}$ couples to gravity through the scalar curvature rather than only *via* the metric, which was typically considered in the literature.

Having obtained the $f(R)$ theory of (54), we investigate the corresponding electric string and magnetic 5-brane solutions. Such p -brane solitons were extensively studied in supergravities. (See *e.g.* [29].) In addition to reviewing these solutions in the $f(R)$ frame, we shall also consider new p -brane solutions that would be discarded in the usual supergravity discussions.

Electric String: Our $f(R)$ theory admits the following electric string solution

$$\begin{aligned} ds^2 &= H^{-\frac{2}{3}}(-dt^2 + dx^2) + H^{\frac{1}{3}}dy^i dy^i, \\ F_{(3)} &= dt \wedge dx \wedge dH^{-1}, \quad \partial_i \partial_i H = 0. \end{aligned} \quad (60)$$

The metric of this solution is simply a certain conformal scaling of the NS-NS string solution. The isotropic one is given by

$$dy^i dy^i = dr^2 + r^2 d\Omega_7^2, \quad H = 1 + q/r^6, \quad (61)$$

Thus we have

$$F_{(3)}^2 = 6H^{-3}H'^2 = \frac{6^3 q^2 r^4}{(r^6 + q)^3} \quad F = H^{-\frac{1}{3}}. \quad (62)$$

In $f(R)$ supergravity, the solution is regular from the horizon $r = 0$, which is $\text{AdS}_{(3)} \times S^7$ to the asymptotic $r = \infty$ flat spacetime. From the second equation (57), we can define the conserved electric string charge

$$Q_1 = \int_{r \rightarrow \infty} F^{-1} * F_{(3)} = 7q\omega_7, \quad (63)$$

where ω_7 is the volume of the unit S^7 . In this paper, we denote ω_n and $\Omega_{(n)}$ as the volume and the volume form of the unit S^n respectively. The “1” in the harmonic function can be

dropped, giving rise to the $\text{AdS}_3 \times S^7$ solution

$$ds^2 = \ell^2 \left(\frac{dr^2}{r^2} + \ell^{-6} r^4 (-dt^2 + dx^2) + d\Omega_{(3)}^2 \right), \quad F_{(3)} = 6\ell^{-6} r^5 dt \wedge dx \wedge dr. \quad (64)$$

It should be pointed out that this solution cannot be obtained by the decoupling limit that is typically considered in the context of the AdS/CFT correspondence. If we scale $r \rightarrow \epsilon r$ and $x^\mu \rightarrow \epsilon^3 x^\mu$, and then send ϵ to zero, we find that although the metric becomes $\text{AdS}_3 \times S^7$, the quantity $F_{(3)}$ blows up. This is related to the fact that quantities $f(R)$ and $F_{(3)}^2$, $F(R)$ in (64) are all divergent in the AdS boundary. However, this does not affect the earlier statement that the string solution interpolates smoothly between the $\text{AdS}_3 \times S^7$ horizon and the asymptotic flat spacetime.

In the $F_{(7)}$ dual description, where $F_{(7)}$ is magnetic, the metric of the solution is identical, but with $F_{(7)} = 6q\Omega_7$. In this case, the $\text{AdS}_{(3)} \times S^7$ can be obtained from the decoupling limit, with both $F_{(7)}^2$ and $f(R)$ finite, if one overlooks the divergence of $F(R)$ in this limit.

If we trace back to $D = 11$, the string solution becomes the usual M2-brane, and the $\text{AdS}_3 \times S^7$ becomes $\text{AdS}_4 \times S^7$. Since half of the Killing spinors in $\text{AdS}_4 \times S^7$ depend on the world-volume coordinate [30], it follows that the $\text{AdS}_3 \times S^7$ preserves half of the supersymmetry, with no supersymmetry enhancement. Further discussion of supersymmetry will be given in section 5.

Magnetic 5-brane: There are two types of magnetic 5-brane solutions. The first type is given by

$$\begin{aligned} ds_{10}^2 &= H^{-\frac{1}{3}} dx^\mu dx_\mu + H^{\frac{2}{3}} (dr^2 + r^2 d\Omega_{(3)}^2), \\ F_{(3)} &= 2q\Omega_{(3)}, \quad H = 1 + \frac{q}{r^2}. \end{aligned} \quad (65)$$

The solution carries the magnetic charge

$$Q_5 = \int F_{(3)} = 4Q\pi^2. \quad (66)$$

This solution is effective the usual NS-NS 5-brane written in the new frame of the $f(R)$ theory. Since $F_{(3)}$ is constantly proportional to $\Omega_{(3)}$, the isometry of S^3 is preserved. The solution however suffers a curvature power-law singularity at $r = 0$. Since the function H is a harmonic function in the transverse space, the solution can be generalized to a solution describing multi-center 5-branes. We can lift the solution back to $D = 11$ and obtain the smeared M5-brane. Such p -brane solutions and their behavior in different frames were discussed extensively in [29].

The theory in fact admits the second type of magnetic 5-brane solutions that were not considered previously:

$$\begin{aligned} ds_{10}^2 &= H^{-\frac{1}{3}} dx^\mu dx_\mu + H^{\frac{2}{3}} (dr^2 + r^2 d\Omega_3^2), & d\Omega_3^2 &= d\theta^2 + \sin^2 \theta d\Omega_2^2 \\ F_{(3)} &= 3q \cos \theta \sin^2 \theta d\theta \wedge \Omega_{(2)}, & H &= 1 + \frac{q}{r^3}, \end{aligned} \quad (67)$$

In contrast to the previous 5-brane solution, the metric in this solution is smooth running from $\text{AdS}_7 \times S^3$ at the $r = 0$ horizon to the asymptotic flat spacetime at $r = \infty$. It should be mentioned that in this solution $F_{(3)}$ is not constant proportional to $\Omega_{(3)}$, the volume form of the S^3 , and the function H is not the harmonic function in the transverse space. Thus, the solution does not generalize to the multi-center harmonic solutions.

Unlike the previous string solution, there is a decoupling limit for which the “1” in the function H can be dropped. This can be done first by scaling $r \rightarrow \epsilon r$ and $x^\mu \rightarrow \sqrt{\epsilon} x^\mu$ and then sending ϵ to zero. The resulting $\text{AdS}_7 \times S^3$ solution is given by

$$ds^2 = \ell^2 \left(\frac{dr^2}{r^2} + r dx^\mu dx_\mu + d\Omega_{(3)}^2 \right), \quad F_{(3)} = 3\ell^3 \cos \theta \sin^2 \theta d\theta \wedge \Omega_{(2)}, \quad (68)$$

Note also that the solution has the following properties

$$F_{(3)} = 3q \cos \theta \Omega_{(3)}, \quad F(R) = (r^3 + q)^{\frac{1}{3}} \cos \theta. \quad (69)$$

The fact that $F_{(3)}$ is not constantly proportional to the S^3 volume form implies that not all the S^3 isometry is preserved; only the $SO(3)$ of S^2 is preserved. Also $F(R)$ vanishes at $\theta = \pi/2$. Thus the solution would have a naked power-law curvature singularity at $\theta = \pi/2$ if it were to be converted to that of the usual $\mathcal{N} = 1$, $D = 10$ supergravity. From the point of view of the $\mathcal{N} = 1$, $D = 10$ supergravity, the coordinate θ can only run from 0 to $\pi/2$ and hence the 5-brane charge is given by $Q_5 = \int F_{(3)} = 4\pi q$.

However, from the point of view in the $f(R)$ theory, the geodesic completeness of the metric requires that the coordinate θ extend to include the region $[\pi/2, \pi]$ as well. This then implies that that $Q_5 = 4\pi q - 4\pi q = 0$. Furthermore, the function $F(R)$ becomes negative in $\theta \in (\pi/2, \pi]$. Such a problem can be averted by introducing a source such that

$$F_{(3)} = 3q |\cos \theta| \Omega_{(3)}, \quad F(R) = (r^3 + q)^{\frac{1}{3}} |\cos \theta|. \quad (70)$$

Having done that, the equation for the form field in (57) is still exactly satisfied; however, the Einstein equations produce a delta-function source at the equator $\theta = \pi/2$. Such a source on the equator is not uncommon in supergravity solutions. The embedding of the $\text{AdS}_6 \times S^4$ of the localized D4/D8-brane [31] in massive type IIA supergravity has a

more serious power-law singularity at the equator of the S^4 . (See section 4.4.) With this set up, the magnetic charge is doubled, namely

$$Q_5 = \int F_{(3)} = 8\pi q. \quad (71)$$

Thus we see that not only the global structures of the metrics in $f(R)$ and original supergravities are different. The magnetic charges are different too and hence there is no reason to claim that these two solutions are equivalent even though they are related locally by the conformal scaling. Note that in the dual $F_{(7)}$ description, the 5-brane carries the electric flux with $F_{(7)} = d^6x \wedge dH^{-1}$.

If we trace back the local solution to $D = 11$, it becomes the usual M5-brane with the metric

$$ds_{11}^2 = ds_{10}^2 + F^2 d\psi^2 = H^{-\frac{1}{3}} dx^\mu dx_\mu + H^{\frac{2}{3}} (dr^2 + r^2 d\Omega_3^2 + r^2 \cos^2 \theta d\psi^2), \quad (72)$$

where ψ , with period 2π , is the internal coordinate. Since F appears in $D = 11$ only as F^2 , the absolute-value sign on F drops. From the eleven-dimensional point of view, the latitude angle θ clearly runs from 0 to $\pi/2$. In the $f(R)$ theory, however, we must extend the θ range to $[0, \pi]$ by introducing a δ -function source. Thus, even if the local solution of the 5-brane is the same as that from $D = 11$, it describes a different physical state from the M5-brane. Thus spectrum of the $f(R)$ theory contains states that may not apparently exist in the narrow picture of M-theory.

Finally we would like to mention that if we take $\kappa = -1$ for the conformal scaling (11), the Lagrangian becomes

$$e^{-1} \mathcal{L}_{10} = \varphi R - \frac{1}{2} \varphi^2 F_{(3)}^2. \quad (73)$$

The corresponding $f(R)$ becomes

$$f(R) = \frac{3R^2}{F_{(3)}^2}. \quad (74)$$

Interestingly, as we shall see later, the $f(R)$ theory associated with the R-R 3-form field strength in type IIB supergravity takes this form.

4.2 $f(R)$ heterotic supergravity

$\mathcal{N} = 1$, $D = 10$ supergravities with additional $E_8 \times E_8$ or $SO(32)$ Yang-Mills fields are the low-energy effective theory of the corresponding heterotic string theories. The bosonic Lagrangian for heterotic supergravity is given by

$$e^{-1} \mathcal{L}_{10} = R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{12} e^{-\phi} F_{(3)}^2 - \frac{1}{4} e^{-\frac{1}{2}\phi} (F_{(2)}^I)^2, \quad (75)$$

where $F_{(2)}^I$ are the field strengths for the Yang-Mills fields. The 3-form field strength satisfies the Bianchi identity

$$dF_{(3)} = \frac{1}{2}F_{(2)}^I \wedge F_{(2)}^I. \quad (76)$$

The conformal transformation (11) with $\kappa = +1$ leaves the Yang-Mills fields decoupled from the scalar, namely

$$e^{-1}\mathcal{L}_{10} = \varphi R - \frac{1}{2}\varphi^{-1}F_{(3)}^2 - \frac{1}{4}(F_{(2)}^I)^2, \quad (77)$$

It follows that the bosonic Lagrangian of the $f(R)$ heterotic supergravity is given by

$$e^{-1}\mathcal{L}_{10} = \sqrt{-\frac{1}{3}RF_{(3)}^2} - \frac{1}{4}(F_{(2)}^I)^2. \quad (78)$$

4.3 (Massive) type IIA $f(R)$ supergravity

Type IIA supergravity [32] is the low-energy effective theory for the type IIA string. In addition to the NS-NS fields that are also present in $\mathcal{N} = 1, D = 10$ supergravity, the bosonic sector includes the R-R vector $\mathcal{A}_{(1)}$ and tensor $A_{(3)}$ as well. The bosonic Lagrangian is

$$e^{-1}\mathcal{L}_{10} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{-\phi}F_{(3)}^2 - \frac{1}{48}e^{\frac{1}{2}\phi}F_{(4)}^2 - \frac{1}{4}e^{\frac{3}{2}\phi}\mathcal{F}_{(2)}^2 + e^{-1}\mathcal{L}_{\text{FFA}}. \quad (79)$$

where

$$\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}, \quad F_{(3)} = dA_{(2)}, \quad F_{(4)} = dA_{(3)} + A_{(1)} \wedge \mathcal{F}_{(2)}. \quad (80)$$

Choosing the $\kappa = +1$ conformal scaling, we have

$$e^{-1}\mathcal{L}_{10} = \varphi(R - \frac{1}{48}F_{(4)}^2) - \frac{1}{4}\varphi^3F_{(2)}^2 - \frac{1}{12}\varphi^{-1}F_{(3)}^2. \quad (81)$$

Thus we see that the $f(R)$ frame is the same as that of the D2-branes. In order to find $f(R)$, we can first find $F(R)$, which satisfies the following polynomial

$$36F_{(2)}^2F^4 + (F_{(4)}^2 - 48R)F^2 - 4F_{(3)}^2 = 0, \quad (82)$$

The $f(R)$ is thus given by

$$\begin{aligned} f &= \frac{1}{432}(2F_{(4)}^2 + X - 96R)\sqrt{\frac{X - F_{(4)}^2 + 48R}{2F_{(2)}^2}}, \\ X^2 &= 576F_{(2)}^2F_{(3)}^2 + (F_{(4)}^2 - 48R)^2. \end{aligned} \quad (83)$$

In general, it is smooth to take various field strength to zero. This can be seen from (82). A special case arises when we consider $F_{(3)} = 0 = \mathcal{F}_{(2)}$, leaving only the $F_{(4)}$ non-vanishing. In this case, the (81) implies that

$$e^{-1}\mathcal{L}_{10} = \varphi(R - \frac{1}{48}F_{(4)}^2), \quad (84)$$

which corresponding to the singular case discussed in section 2. From the point of view of (83), it is a singular limit by letting $F_{(3)}$ and $\mathcal{F}_{(2)}$ vanish simultaneously, and the regularity requires us to introduce a scalar proportional to F in order to keep the theory regular, leading to (84).

The harmonic electric D2-brane can also be easily constructed. We shall present only the isotropic solution

$$\begin{aligned} ds_{10}^2 &= H^{-\frac{2}{3}} dx^\mu dx_\mu + H^{\frac{1}{3}} (dr^2 + r^2 d\Omega_6^2), \\ \varphi &= H^{\frac{1}{6}}, \quad F_{(4)} = d^3 x \wedge dH^{-1}, \quad H = 1 + \frac{q}{r^5}. \end{aligned} \quad (85)$$

It is clear that the solution suffers from a power-law curvature singularity at $r = 0$, as in the case in the original Einstein frame.

There exists an alternative non-harmonic D2-brane

$$\begin{aligned} ds_{10}^2 &= H^{-\frac{2}{3}} dx^\mu dx_\mu + H^{\frac{1}{3}} (dr^2 + r^2 d\Omega_6^2), \\ \varphi &= (r^6 + q)^{\frac{1}{6}} |\cos \theta|, \quad F_{(4)} = d^3 x \wedge dH^{-1}, \quad H = 1 + \frac{q}{r^6}, \\ d\Omega_6^2 &= d\theta^2 + \sin^2 \theta d\Omega_5^2 \end{aligned} \quad (86)$$

The electric charge is given by

$$Q_2 = \int_{r \rightarrow \infty} \varphi F_{(4)} = \frac{6\omega_7}{\pi} q. \quad (87)$$

The discussion of the properties of this solution is analogous to that of the string solution in the previous subsection, and hence we shall not elaborate further.

The Lagrangian admits a magnetic D4-brane solution, given by

$$\begin{aligned} ds_{10}^2 &= H^{-\frac{1}{3}} dx^\mu dx_\mu + H^{\frac{2}{3}} (dr^2 + r^2 d\Omega_4^2), \\ \varphi &= H^{-\frac{1}{6}}, \quad F_{(4)} = 3q\Omega_{(4)}, \quad H = 1 + \frac{q}{r^3}. \end{aligned} \quad (88)$$

where $\Omega_{(4)}$ is the volume form for the unit S^4 . Thus the solution interpolates between $\text{AdS}_6 \times S^4$ at the horizon $r = 0$ to the asymptotic flat spacetime at $r = \infty$. The quantity $F_{(4)}^2$ is clearly finite in this region. Furthermore, the scalar φ is also finite, running from zero 0 to 1. From the point of view of $f(R)$ theory, the φ is simply F and hence the whole solution should be viewed as regular. This interpretation is very different from the D4-brane in the usual type IIA supergravity. Note that here H is the harmonic function in the transverse space, and hence solution can be generalized to describe multi-center D4-branes. The $\text{AdS}_6 \times S^4$ solution can be obtained by dropping the “1” in the function H , namely

$$ds_{10}^2 = \ell^2 \left(\frac{dr^2}{r^2} + \ell^{-3} r dx^\mu dx_\mu + d\Omega_{(4)}^2 \right),$$

$$F_{(4)} = 3\ell^3 \Omega_{(4)}, \quad F = \sqrt{\frac{r}{\ell}} \quad (89)$$

Note that the proper coordinate for the AdS space is $r = \ell\rho^2$. It follows that r will not go negative. In terms of the coordinate ρ , the D4-brane metric can be viewed as a symmetric worm-3brane separated by a bulk horizon. The positiveness of φ requires that $\varphi \equiv F = |\rho|$ and hence the solution requires a delta function source. This is analogous to the type IIB instanton solution [34], which is a wormhole in the string frame. The Einstein equations of motion however involves $\square H$ where H is the harmonic function on ten-dimensional Euclidean space. Thus a matter source is also needed.

Lifting the D4-brane to $D = 11$ gives rise to the standard isotropic M5-brane, in which case the solution is totally regular and there is no need for any source. This is because only $r = \rho^2$ appears in the metric and hence the sign choice of ρ is irrelevant. For this reason one can identify the inside with the outside of the M5-brane [33]. The different interpretation suggests that the $f(R)$ type IIA supergravity contains states that are outside the physical spectrum in M-theory.

The NS-NS string and 5-brane were already presented in the previous subsectoin. D0-brane and D6-brane solutions are given by

$$\begin{aligned} D0 : \quad & ds^2 = -H^{-1}dt^2 + dr^2 + r^2 d\Omega_8^2, \\ & \mathcal{F}_{(2)} = dt \wedge dH^{-1}, \quad H = 1 + \frac{q}{r^7}, \\ D6 : \quad & ds^2 = dx^\mu dx_\mu + H(dr^2 + r^2 d\Omega_{(2)}^2), \\ & \mathcal{F}_{(2)} = q\Omega_{(2)}, \quad H = 1 + \frac{q}{r}. \end{aligned} \quad (90)$$

Both solutions are singular at $r = 0$.

Massive type IIA supergravity in ten dimensions were constructed in [17]. After the conformal scaling (11), the bosonic Lagrangian is given by

$$e^{-1}\mathcal{L}_{10} = \varphi(R - \frac{1}{48}F_{(4)}^2) - \frac{1}{12}\varphi^{-1}F_{(4)}^2 - \frac{1}{4}\varphi^3\mathcal{F}_{(2)}^2 - \frac{1}{2}m^2\varphi^5 + e^{-1}\mathcal{L}_{FFA}, \quad (91)$$

where

$$\begin{aligned} \mathcal{F}_{(2)} &= dA_{(1)} + mA_{(2)}, \quad F_{(3)} = dA_{(2)}, \\ F_{(4)} &= dA_{(3)} + \mathcal{A}_{(1)} \wedge dA_{(2)} + \frac{1}{2}A_{(2)} \wedge A_{(2)}, \\ \mathcal{L}_{FFA} &= \Sigma_{(10)}, \quad \text{with} \quad d\Sigma_{(10)} = -\frac{1}{2}F_{(4)} \wedge F_{(4)} \wedge F_{(3)}. \end{aligned} \quad (92)$$

The essence of this theory is that the NS-NS 2-form eats the R-R vector fields and becomes massive. The $\varphi \equiv F(R)$ function satisfies the following polynomial equation

$$\frac{5}{2}m^2F^6 + \frac{3}{4}\mathcal{F}_{(2)}^2F^4 + (\frac{1}{48}F_{(4)}^2 - R)F^2 + \frac{1}{12}F_{(3)}^2 = 0. \quad (93)$$

Whilst the analytic form for F exists in this case, it is not instructive to give explicitly here. One extra ingredient in massive type IIA theory is the exponential scalar potential associated with the D8-brane, where m is the 8-brane charge. The corresponding $f(R)$ description is

$$f(R) = \frac{4}{5} \left(\frac{2}{5m^2} \right)^{\frac{1}{4}} R^{\frac{5}{4}}, \quad F(R) = \left(\frac{2}{5m^2} \right)^{\frac{1}{4}} R^{\frac{1}{4}}. \quad (94)$$

Note that we have also presented the $F(R)$ here. Thus for this $f(R)$ theory, any metric with vanishing Ricci scalar R is a solution, since $f(0) = 0 = F(0)$. Since the equations of motion is reduced to a scalar-like equation $R = 0$, there is no propagating spin-2 degrees of freedom in this background. (See *e.g.* [10].) This solution does not exist in the original massive type IIA supergravity and it cannot be lifted to $D = 11$ either.

4.4 Type IIB $f(R)$ supergravity

Type IIB supergravity was constructed in [35] at the level of equations of motion since there can be no Lagrangian formalism for the self-dual 5-form field strength. It is nevertheless possible to write a Lagrangian with a non-self-dual 5-form and then impose the self-duality by hand after deriving the equations of motion from the Lagrangian [36]. After the conformal scaling (11), the bosonic Lagrangian of the type IIB supergravity becomes

$$e^{-1} \mathcal{L} = \varphi R - \frac{1}{12} \varphi^{-1} (F_{(3)}^{\text{NS}})^2 - \frac{1}{2} \varphi^4 (\partial \chi)^2 - \frac{1}{12} \varphi^2 (F_{(3)}^{\text{RR}})^2 - \frac{1}{240} F_{(5)}^2 + e^{-1} \mathcal{L}_{\text{FFA}}, \quad (95)$$

where

$$\begin{aligned} F_{(3)}^{\text{NS}} &= dA_{(2)}^{\text{NS}}, & F_{(3)}^{\text{RR}} &= dA_{(2)}^{\text{RR}} - \chi dA_{(2)}^{\text{NS}}, \\ dF_{(5)} &= F_{(3)}^{\text{NS}} \wedge F_{(3)}^{\text{RR}}, & \mathcal{L}_{\text{FFA}} &= \frac{1}{2} A_{(4)} \wedge dA_{(2)}^{\text{NS}} \wedge dA_{(2)}^{\text{RR}}. \end{aligned} \quad (96)$$

Here we adopt the notation of [37]. The self-duality condition for the 5-form should be imposed at the level of equations of motion [36]. Unfortunately, there can be no analytical $f(R)$ description of type IIB supergravity since the function F satisfies an equation of the quintic-order polynomials. The $f(R)$ theory for the NS-NS 3-form was given earlier. The one for each individual R-R fields will be given in the next subsection. Here we shall draw attention to the fact that the $f(R)$ theory for the R-R 3-form is identical to that in (74), which has an origin from $D = 11$ as the $\kappa = -1$ reduction.

4.5 Further discussions on the R-R sector

The $D(p-1)$ -branes in supergravities are brane solutions supported by the R-R p -form field strength. In general, the relevant bosonic Lagrangian, under the conformal scaling (11), is

$$e^{-1}\mathcal{L} = \varphi R - \frac{1}{2p!}\varphi^{5-p}F_p^2. \quad (97)$$

The associated $f(R)$ is

$$f(R) = \left(\frac{2p!}{5-p}\right)^{\frac{1}{4-p}} \frac{p-4}{p-5} R \left(\frac{R}{F_p^2}\right)^{\frac{1}{4-p}}, \quad \text{with} \quad f = \frac{p-4}{p-5} R F. \quad (98)$$

Thus the $f(R)$ description appears to break down for $p = 4$ and 5 , corresponding to D4, D6 and D5 branes. However, this only implies that when we take a limit to $p = 4$ and 5 cases from the full general $f(R)$ theory, a proper care should be taken. Let us consider the D4-D8 system as an example, which involves both the $F_{(4)}$ and the scalar potential in the massive type IIA supergravity. The relevant Lagrangian is

$$e^{-1}\mathcal{L} = \varphi(R - \frac{1}{48}F_{(4)}^2) - \frac{1}{2}m^2\varphi^5. \quad (99)$$

The $f(R)$ theory is given by

$$f(R) = \frac{4}{5}F(R) \left(R - \frac{1}{48}F_{(4)}^2\right), \quad F = \left(\frac{2}{5m^2}(R - \frac{1}{48}F_{(4)}^2)\right)^{\frac{1}{4}}. \quad (100)$$

The second equation above implies that there is a smooth limit with $m \rightarrow 0$ that recovers (99) modulo an overall factor $\frac{4}{5}$.

It was shown that [38, 39] AdS_6 arises in the localized D4/D8-brane system [31], and the gauged AdS_6 supergravity [24] can be obtained from spherical reduction from massive type IIA supergravity [39]. The relevant $f(R)$ theory for the D4/D8 system is given by above. The AdS_6 embedding in the $f(R)$ theory becomes

$$\begin{aligned} ds_{10}^2 &= (\cos\theta)^{\frac{2}{9}}(ds_{\text{AdS}_6}^2 + 2d\theta^2 + 2\sin^2\theta d\Omega_3^2), \\ F_{(4)} &= \frac{5\sqrt{2}}{6}(\cos\theta)^{\frac{1}{3}}\sin^3\theta d\theta \wedge \Omega_{(3)}, \quad F = (\cos\theta)^{-\frac{5}{9}}. \end{aligned} \quad (101)$$

Thus the solution has a power-law curvature singularity at the equator $\theta = \pi/2$ of the S^4 . The solution becomes regular in the D4-brane frame [40]. In the D4-brane, the Lagrangian for massive type IIA supergravity has the form

$$e^{-1}\mathcal{L} = \tilde{\varphi}(R - 2(\partial\log\tilde{\varphi})^2 - \frac{1}{1440}F_{(6)}^2) + \text{more}. \quad (102)$$

The metric of the D4/D8 solution is then simply the $\text{AdS}_6 \times S^4$ without the pre-factor in (101) and furthermore, $\tilde{\varphi} = (\sin\theta)^{1/3}$. Thus it is necessary that θ runs from 0 to π with the

$(\cos \theta)^{1/3}$ factor in both $F_{(4)}$ and $\tilde{\varphi}$ added an absolute-value sign, namely $|(\cos \theta)^{1/3}|$. This requires a delta-function source on the equator. This is very analogous to the new solutions of ten-dimensional $f(R)$ supergravities we have obtained.

Before we end this section we would like to remark that in all supergravities in ten dimensions, the algebraic equation associated with the dilaton φ in the $f(R)$ frame are polynomials of integer power. This property is not universal. In section 8, we present an example of $\mathcal{N} = 1$, $D = 7$ gauged supergravity, and the resulting polynomial is of irrational powers. This suggests that ten-dimensional supergravities are special from the point of view of the $f(R)$ formalism. Indeed ten-dimensional supergravities play two roles in string and M-theory. One is that they are the low-energy effective theories of string; the other is that they are related to M-theory through the dimensional compactification.

5 $f(R)$ supergravities: the fermionic sector

In the previous section, we demonstrate that the bosonic sectors of supergravities in ten dimensions can be converted to $f(R)$ theories coupled to the form fields. In this section, we shall consider the fermion fields and show that such a conversion extended to the fermionic sector, at least up to the quadratic order in fermions. The key point is that the dilaton in the $f(R)$ frame remains auxiliary even when the fermionic sector is included.

5.1 A general discussion

All the supergravity theories considered in the previous section contain a dilaton that is part of the supergravity multiplet. The truncated Lagrangian involving the metric, dilaton, gravitino and dilatino takes the universal form

$$e^{-1}\mathcal{L}_D = R - \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho + \frac{1}{2}\bar{\lambda}\Gamma^\mu D_\mu\lambda + \frac{i}{2\sqrt{2}\beta}\bar{\lambda}\Gamma^\mu\Gamma^\nu\psi_\mu\partial_\nu\phi, \quad (103)$$

where $\beta = \pm 1$ depending on the fermion convention presented in Table 1 below. (This β has nothing to do with the constant defined in (12).) The relevant parts of the supersymmetric transformation rules are given by

$$\begin{aligned} \delta\psi_\mu &= D_\mu\epsilon, & \delta\lambda &= \frac{i\sqrt{\beta}}{2\sqrt{2}}\Gamma^\mu\partial_\mu\phi\epsilon, \\ \delta e_\mu^a &= \frac{1}{4}\bar{\psi}_\mu\Gamma^a\epsilon, & \delta\phi &= \frac{i}{2\sqrt{2}\beta}\bar{\epsilon}\lambda. \end{aligned} \quad (104)$$

It should be pointed out that the Lagrangian (103) is already invariant under the supersymmetric transformation rules up to the quadratic order in fermions, and hence the theory can be viewed as pseudo-supergravity discussed in [14, 15].

We first make the following conformal transformation and field redefinition,

$$\begin{aligned} e_\mu^a &= \varphi^{\frac{1}{D-2}} \tilde{e}_\mu^a, & \psi_\mu &= \varphi^{\frac{1}{2(D-2)}} \left(\tilde{\psi}_\mu - \frac{i}{\sqrt{(D-1)(D-2)\beta}} \Gamma_\mu \tilde{\lambda} \right), \\ \lambda &= \varphi^{-\frac{1}{2(D-2)}} \tilde{\lambda}, & \epsilon &= \varphi^{\frac{1}{2(D-2)}} \tilde{\epsilon}, \end{aligned} \quad (105)$$

where φ is related to ϕ by (11) and (12) with $\kappa = +1$. Substituting these into (103) and then dropping the tilde, we find that the Lagrangian becomes

$$\begin{aligned} e^{-1} \mathcal{L}_D &= \varphi \hat{R}, & \hat{R} &= R + K, \\ K &= \frac{1}{2} \bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho - i \sqrt{\frac{(D-2)\beta}{D-1}} \bar{\lambda} \Gamma^{\mu\nu} D_\mu \psi_\nu + \frac{1}{2} D^\nu (\bar{\psi}_\mu \Gamma^\mu \psi_\nu). \end{aligned} \quad (106)$$

The supersymmetric transformation rules become

$$\begin{aligned} \delta \psi_\mu &= D_\mu \epsilon, & \delta \lambda &= \frac{i \sqrt{(D-1)\beta}}{2\sqrt{D-2}} \varphi^{-1} \Gamma^\mu \partial_\mu \varphi \epsilon, \\ \delta e_\mu^a &= \frac{1}{4} \bar{\psi}_\mu \Gamma^a \epsilon, & \delta \varphi &= -\frac{i \sqrt{(D-2)\beta}}{4\sqrt{D-1}} \bar{\lambda} \epsilon. \end{aligned} \quad (107)$$

Note that in converting the theory to the new frame, δe_μ^a will generate a local Lorentz transformation which we have dropped. Thus we see that in the new $f(R)$ frame where the dilaton loses its kinetic term, the contribution of the fermions is simply to replace Ricci scalar with \hat{R} , with φ remaining auxiliary. Thus we have the generic result that for this contribution: we can simply replace the previous $f(R)$ with $f(\hat{R})$.

5.2 $\mathcal{N} = 1$, $D = 10$ $f(R)$ supergravity

The full $\mathcal{N} = 1$, $D = 10$ supergravity theory was constructed in [28]. In this paper, we shall consider the fermionic sector only up to the quadratic order in fermion fields. The supersymmetric partners are gravitino ψ_μ and dilitino λ , which are both Majorana and Weyl. Making the field redefinition (105) and then drop the tilde, we find that the Lagrangian in the $f(R)$ frame is given by

$$e^{-1} \mathcal{L} = \varphi (R + K) - \frac{1}{12} \varphi^{-1} F_{(3)}^2 + X_3, \quad (108)$$

where K and the Yukawa term X_3 associated with $F_{(3)}$ are given by

$$\begin{aligned} K &= \frac{1}{2} \bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{2\sqrt{2}i}{3} \bar{\lambda} \Gamma^{\mu\nu} D_\mu \psi_\nu + \frac{1}{2} D^\mu (\bar{\psi}_\nu \Gamma^\nu \psi_\mu), \\ X_3 &= \left(-\frac{1}{48} \bar{\psi}_\mu \Gamma^{\mu\nu\rho\sigma\lambda} \psi_\lambda - \frac{1}{8} \bar{\psi}^\nu \Gamma^\rho \psi^\sigma + \frac{\sqrt{2}i}{12} \bar{\lambda} \Gamma^{\nu\rho} \psi^\sigma \right) F_{\nu\rho\sigma}. \end{aligned} \quad (109)$$

The supersymmetric transformation rules in the $f(R)$ frame are given by

$$\delta \psi_\mu = D_\mu \epsilon + \frac{1}{54} F_{\nu\rho\sigma} \Gamma_\mu^{\nu\rho\sigma} \epsilon - \frac{1}{12} F_{\mu\nu\rho} \Gamma^{\nu\rho} \epsilon,$$

$$\begin{aligned}
\delta\lambda &= \frac{3i}{4\sqrt{2}}\varphi^{-1}(\Gamma^\mu\partial_\mu\varphi - \frac{1}{18}F_{\mu\nu\rho}\Gamma^{\mu\nu\rho})\epsilon, \\
\delta e_\mu^a &= \frac{1}{4}\bar{\psi}_\mu\Gamma^a\epsilon, \quad \delta g_{\mu\nu} = \frac{1}{2}\bar{\psi}_{(\mu}\gamma_{\nu)}\epsilon, \quad \delta\varphi = -\frac{i}{3\sqrt{2}}\varphi\bar{\lambda}\epsilon, \\
\delta A_{\mu\nu} &= \varphi(-\frac{1}{2}\bar{\epsilon}\Gamma_{[\mu}\psi_{\nu]} + \frac{i}{3\sqrt{2}}\bar{\epsilon}\Gamma_{\mu\nu}\lambda)
\end{aligned} \tag{110}$$

Integrating out φ , we find that the $f(R)$ theory of the $\mathcal{N} = 1$, $D = 10$ $f(R)$ supergravity is given by

$$\begin{aligned}
e^{-1}\mathcal{L} &= \sqrt{-\frac{1}{3}(R+K)F_{(3)}^2} + X_3, \\
\delta\psi_\mu &= D_\mu\epsilon + \frac{1}{54}F_{\nu\rho\sigma}\Gamma_\mu^{\nu\rho\sigma}\epsilon - \frac{1}{12}F_{\mu\nu\rho}\Gamma^{\nu\rho}\epsilon, \\
\delta\lambda &= \frac{3i}{4\sqrt{2}}F^{-1}(\Gamma^\mu\partial_\mu F - \frac{1}{18}F_{\mu\nu\rho}\Gamma^{\mu\nu\rho})\epsilon, \\
\delta e_\mu^a &= \frac{1}{4}\psi_\mu\Gamma^a\epsilon, \quad \delta g_{\mu\nu} = \frac{1}{2}\bar{\psi}_{(\mu}\gamma_{\nu)}\epsilon_0, \\
\delta A_{\mu\nu} &= F(-\frac{1}{2}\bar{\epsilon}\Gamma_{[\mu}\psi_{\nu]} + \frac{i}{3\sqrt{2}}\bar{\epsilon}\Gamma_{\mu\nu}\lambda)
\end{aligned} \tag{111}$$

One can add further the matter Yang-Mills multiplet (A_μ, χ) . In the $f(R)$ frame, the extra parts of the Lagrangian and supersymmetric transformation rules are given by

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{YM}} &= -\frac{1}{4}F_{(2)}^2 + \frac{1}{2}\varphi\bar{\chi}\Gamma^\mu D_\mu\chi - \frac{1}{48}F_{\mu\nu\rho}\bar{\chi}\Gamma^{\mu\nu\rho}\chi - \frac{1}{4\sqrt{2}}\varphi^{\frac{1}{2}}F_{\nu\rho}\bar{\chi}\Gamma^\mu\Gamma^{\nu\rho}\psi_\mu, \\
\delta\chi &= \frac{1}{4\sqrt{2}}\Gamma^{\mu\nu}F_{\mu\nu}\epsilon, \quad \delta A_\mu = -\frac{1}{2\sqrt{2}}\varphi^{\frac{1}{2}}\bar{\chi}\Gamma_\mu\epsilon, \\
\delta_{\text{extra}}A_{\mu\nu} &= \frac{1}{2\sqrt{2}}\varphi^{\frac{1}{2}}\bar{\chi}A_{[\mu}\Gamma_{\nu]}\epsilon.
\end{aligned} \tag{112}$$

It is again straightforward to integrate out the auxiliary φ and obtain the $f(R)$ theory of heterotic supergravity.

5.3 Type IIA $f(R)$ supergravity

Type IIA supergravity was constructed in [32]. The superpartners include the Majorana gravitino and dilatino. We first convert the the mostly minus convention of the spacetime signature of [32] to the mostly plus one. Making an analogous field reduction (105) with appropriate Γ_{11} inserted, we find that the Lagrangian of type IIA supergravity in the $f(R)$ frame up to the quadratic fermionic order is given by

$$\begin{aligned}
e^{-1}\mathcal{L}_{10} &= e^{-1}\mathcal{L}_B + \varphi\left(\frac{1}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{2\sqrt{2}i}{3}\bar{\lambda}\Gamma^{\mu\nu}\Gamma_{11}D_\mu\psi_\nu + \frac{1}{2}D^\mu(\psi_\nu\Gamma^\nu\psi_\mu)\right) \\
&+ \left(-\frac{1}{48}\bar{\psi}_\mu\Gamma_{11}\Gamma^{\mu\nu\rho\sigma}\psi_\lambda - \frac{1}{8}\psi^\nu\Gamma_{11}\Gamma^\rho\psi^\sigma + \frac{\sqrt{2}i}{12}\bar{\lambda}\Gamma_{11}\Gamma^{\nu\rho}\psi^\sigma\right)F_{\nu\rho\sigma} \\
&+ \varphi^2\left(\frac{1}{16}\bar{\psi}_\alpha\Gamma^{\alpha\beta\mu\nu}\psi_\beta + \frac{1}{8}\bar{\psi}^\mu\psi^\nu + \frac{\sqrt{2}}{24}\bar{\lambda}\Gamma^\alpha\Gamma^{\mu\nu}\psi_\alpha + \frac{\sqrt{2}}{6}\bar{\lambda}\Gamma^\mu\psi^\nu - \frac{1}{144}\bar{\lambda}\Gamma_{11}\Gamma^{\mu\nu}\lambda\right)F_{\mu\nu} \\
&+ \varphi\left(-\frac{1}{192}\bar{\psi}_\alpha\Gamma^{\alpha\mu\nu\rho\sigma\beta}\psi_\beta - \frac{1}{16}\bar{\psi}^\mu\Gamma^{\nu\rho}\psi^\sigma + \frac{1}{72\sqrt{2}}\bar{\lambda}\Gamma_{11}\Gamma^{\mu\nu\rho\sigma\alpha}\psi_\alpha\right)F_{\mu\nu\rho\sigma}.
\end{aligned} \tag{113}$$

where \mathcal{L}_B is given in (81). We see that at least up to the quadratic fermion order, the scalar φ is auxiliary and can be integrated out, yielding an $f(R)$ theories of type IIA supergravity.

Furthermore, there is no derivative on the dilatino λ in the Lagrangian and hence it is also an auxiliary field and can be integrated out.

The supersymmetric transformation rules in the $f(R)$ frame up to quadratic order in fermions are given by

$$\begin{aligned}
\delta\psi_\mu &= D_\mu\epsilon - \left(\frac{1}{288}\Gamma_\mu^{\nu\rho\sigma\lambda} - \frac{1}{36}\delta_\mu^\nu\Gamma^{\rho\sigma\lambda}\right)F_{\nu\rho\sigma\lambda}\epsilon - \varphi\left(\frac{1}{32}\Gamma_\mu^{\nu\rho} - \frac{3}{10}\delta_\mu^\nu\Gamma^\rho\right)\Gamma_{11}F_{\nu\rho}\epsilon \\
&\quad - \varphi^{-1}\left(\frac{1}{72}\Gamma_\mu^{\nu\rho\sigma} - \frac{1}{12}\delta_\mu^\nu\Gamma^{\rho\sigma}\right)\Gamma_{11}F_{\nu\rho\sigma}\epsilon, \\
\delta\lambda &= -\frac{3}{4\sqrt{2}}\left(\Gamma^\mu\Gamma_{11}\varphi^{-1}\partial_\mu\varphi - \frac{1}{144}\Gamma^{\mu\nu\rho\sigma}\Gamma_{11}F_{\mu\nu\rho\sigma} - \frac{1}{18}\varphi^{-1}\Gamma^{\mu\nu\rho}F_{\mu\nu\rho} + \frac{1}{4}\varphi\Gamma^{\mu\nu}F_{\mu\nu}\right)\epsilon, \\
\delta e_\mu^a &= \frac{1}{4}\bar{\psi}_\mu\Gamma^a\epsilon, \quad \delta g_{\mu\nu} = \frac{1}{2}\bar{\psi}_{(\mu}\gamma_{\nu)}\epsilon, \quad \delta\varphi = -\frac{1}{3\sqrt{2}}\varphi\bar{\lambda}\Gamma_{11}\epsilon, \\
\delta A_\mu &= -\frac{1}{4}\varphi^{-1}\left(\bar{\psi}_\mu\Gamma_{11} - \frac{2\sqrt{2}}{3}\bar{\lambda}\Gamma_\mu\right)\epsilon, \\
\delta A_{\mu\nu} &= -\frac{1}{2}\varphi\left(\bar{\psi}_{[\mu}\Gamma_{\nu]} - \frac{\sqrt{2}}{3}\bar{\lambda}\Gamma_{\mu\nu}\right)\epsilon, \\
\delta A_{\mu\nu\rho} &= -\frac{3}{4}\bar{\psi}_{[\mu}\Gamma_{\nu\rho]}\epsilon + 3A_{[\mu}\delta A_{\nu\rho]}. \tag{114}
\end{aligned}$$

6 $f(R)$ gauged (pseudo) supergravities

In this section, we demonstrate that some gauged supergravities can be converted into $f(R)$ supergravities. We then demonstrate that the $f(R)$ theories admit new solutions with global properties that do not exist in the corresponding solutions of the original theories.

6.1 $\mathcal{N} = 2$, $D = 5$ gauged $f(R)$ supergravity

The minimum supergravity in $D = 5$ has $\mathcal{N} = 2$ supersymmetry from the counting scheme in $D = 4$. It contains the metric, a Maxwell vector and a gravitino that is a symplectic majorana. The theory can be gauged with the symplectic structure broken down to the $U(1)$ symmetry. (See *e.g.* [41].) The theory with an arbitrary number of vector multiplet was constructed in [42]. Here, we shall consider only one vector multiplet which consists of a scalar, a vector and a dilatino. The Lagrangian for the bosonic sector is given by

$$\begin{aligned}
e^{-1}\mathcal{L}_5 &= R - \frac{1}{2}(\partial\phi)^2 + 4g^2\left(2e^{-\frac{1}{\sqrt{6}}\phi} + e^{\frac{2}{\sqrt{6}}\phi}\right) - \frac{1}{4}e^{-\frac{2}{\sqrt{6}}\phi}F_{(2)}^2 - \frac{1}{4}e^{\frac{4}{\sqrt{6}}\phi}\mathcal{F}_{(2)}^2 \\
&\quad + \frac{1}{8}e^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}\mathcal{A}_\lambda, \tag{115}
\end{aligned}$$

where $F_{(2)} = dA_{(1)}$ and $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$. The supersymmetric transformation rules are given by

$$\begin{aligned}
\delta\psi_\mu &= [D_\mu - \frac{i}{2}g(\sqrt{2}A_\mu + \mathcal{A}_\mu)]\epsilon + \frac{1}{6}g(2e^{\frac{1}{\sqrt{6}}\phi} + e^{-\frac{2}{\sqrt{6}}\phi})\epsilon \\
&\quad + \frac{i}{8}(\Gamma_\mu^{\nu\rho} - 4\delta_\mu^\nu\Gamma^\rho)(\sqrt{2}e^{-\frac{1}{\sqrt{6}}\phi}F_{\nu\rho} + e^{\frac{2}{\sqrt{6}}\phi}\mathcal{F}_{\nu\rho})\Gamma_\mu\epsilon, \\
\delta\lambda &= -\frac{i}{4}\Gamma^\mu\partial_\mu\phi\epsilon + \frac{i}{\sqrt{6}}g(e^{\frac{1}{\sqrt{6}}\phi} - e^{-\frac{2}{\sqrt{6}}\phi})\epsilon \\
&\quad - \frac{3}{8\sqrt{6}}(\sqrt{2}e^{-\frac{1}{\sqrt{6}}\phi}F_{\mu\nu} - 2e^{\frac{2}{\sqrt{6}}\phi}\mathcal{F}_{\mu\nu})\Gamma^{\mu\nu}\epsilon, \tag{116}
\end{aligned}$$

We have also consulted the papers [43, 44, 23] in deriving the above results. If we set $g = 0$, the theory becomes the ungauged $\mathcal{N} = 2$ supergravity in $D = 5$ with a vector multiplet. The ungauged theory can be obtained from the S^1 reduction [22] of the $\mathcal{N} = (1, 0)$, $D = 6$ supergravity. The vector $\mathcal{A}_{(1)}$ is simply the Kaluza-Klein vector, and $A_{(1)}$ has an origin of the self-dual 3-form field strength in $D = 6$.

Under the conformal scaling (11), the Lagrangian (115) becomes

$$e^{-1}\mathcal{L} = \varphi(R + 8g^2) + \varphi^3(4g^2 - \frac{1}{4}\mathcal{F}_{(2)}^2) - \frac{1}{4}\varphi^{-1}F_{(2)}^2 - \frac{1}{4}e^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}\mathcal{A}_\lambda. \quad (117)$$

Note that apart from the FFA and $F_{(3)}$ term, this Lagrangian is the same as (49) in $D = 5$. It should be reminded that it is typical that the Lagrangians of supergravities in the $f(R)$ frame are polynomials of φ with irrational powers. Whilst all ten-dimensional supergravities have integer powers, such exceptions are rare in lower dimensions.

Performing the following field redefinition for the fermions:

$$\psi_\mu = \varphi^{-\frac{1}{6}}(\tilde{\psi}_\mu + \frac{i}{\sqrt{6}}\Gamma_\mu\tilde{\lambda}), \quad \lambda = \varphi^{-\frac{1}{6}}\tilde{\lambda}, \quad \epsilon = \varphi^{\frac{1}{6}}\tilde{\epsilon}, \quad (118)$$

we find that the supersymmetric transformation rules, after dropping the tilde, become

$$\begin{aligned} \delta\psi_\mu &= [D_\mu - \frac{i}{2}g(\sqrt{2}A_\mu + \mathcal{A}_\mu)]\epsilon + \frac{1}{3}g\varphi\Gamma_\mu\epsilon \\ &\quad + \frac{3\sqrt{2}i}{16}\varphi^{-1}F_{\nu\rho}\Gamma_\mu\Gamma^{\nu\rho}\epsilon - \frac{3i}{4}(\sqrt{2}\varphi^{-1}F_{\mu\nu} + \varphi\mathcal{F}_{\mu\nu})\Gamma^\nu\epsilon, \\ \delta\lambda &= -\frac{i}{\sqrt{6}}\left(\varphi^{-1}\Gamma^\mu\nabla_\mu\varphi - g(\varphi - \varphi^{-1}) - \frac{3i}{8}(\sqrt{2}\varphi^{-1}F_{\mu\nu} - 2\varphi\mathcal{F}_{\mu\nu})\Gamma^{\mu\nu}\right)\epsilon. \end{aligned} \quad (119)$$

Thus the scalar φ in (117) is auxiliary, and can be integrated out straightforwardly, giving to the $f(R)$ theory of gauged supergravity. The algebraic equation for φ^2 is a second-order polynomial and hence can be solved explicitly.

Since our $f(R)$ gauged supergravity is derived from the previously-known $\mathcal{N} = 2$, $D = 5$ gauged supergravity with a vector multiplet, the result is interesting only if the $f(R)$ supergravity is not equivalent to the original theory. This can be seen by examining the solution space of the two theories. Note that in the absence of the Maxwell fields, the $f(R)$ theory is identical to the second example discussed in section 2 with $D = 5$. It follows that this theory admits the following AdS worm-brane solution

$$ds_5^2 = dr^2 + \cosh^2(gr)(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2), \quad F = \tanh(g|r|). \quad (120)$$

The solution is BPS, preserving $\frac{1}{2}$ of the supersymmetry, with Killing spinor given by

$$\tilde{\epsilon} = \cosh^{\frac{1}{2}}(gr)\epsilon_0, \quad \Gamma_r\epsilon_0 = \epsilon_0. \quad (121)$$

One should not view the worm-brane solution (121) simply as a conformal scaling of some domain walls in the original theory. This is because the Ricci scalar R runs from $R = -20g^2$ at the two $r \rightarrow \pm\infty$ asymptotic boundaries to $R = -8g^2$ at $r = 0$, at which point $F = 0$. Since F is the conformal scaling factor between the two theories, the connection between the two theories breaks down for solutions with vanishing or divergent F . The absolute-value sign appearing in F implies that some delta-function matter source is needed for sustaining the wormhole.

One may argue that the power-law curvature singularity of this solution at $r = 0$ of the original gauged supergravity theory is merely an artefact of dimensional reduction since the theory can be embedded in type IIB supergravity. Using the reduction ansatz given in [23], we find that the ten-dimensional metric is

$$\begin{aligned} ds_{10}^2 &= \sqrt{\Delta} \left(dr^2 + \cosh^2(gr) (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + g^{-2} d\theta^2 \right) \\ &\quad + \frac{1}{g^2 \sqrt{\Delta}} \left(\sin^2 \theta d\Omega_3^2 + \tanh^2(gr) \cos^2 \theta d\phi^2 \right), \\ \Delta &= \cos^2 \theta + \tanh^2(gr) \sin^2 \theta. \end{aligned} \quad (122)$$

In $D = 10$, the coordinate r runs from ∞ to 0, without having to go to the negative values of r . This is because when $r \rightarrow 0$, the metric becomes

$$ds_{10}^2 = \cos \theta \left(dr^2 + r^2 d\phi^2 + dx^\mu dx_\mu + g^{-2} d\theta^2 + g^{-2} \tan^2 \theta d\Omega_3^2 \right). \quad (123)$$

The solution does have a power-law curvature singularity when r and $\cos \theta$ vanish simultaneously. Thus we see that the solution behaves differently in $D = 11$ and in $D = 5$. In the $D = 5$ $f(R)$ theory, it is natural for the coordinate r to run from $-\infty$ to $+\infty$, whilst in $D = 10$, it runs from 0 to ∞ .

In [44], non-extremal black hole solutions of the $U(1)^3$ theory in $D = 5$ was constructed. Setting the two of the three vectors equal gives rise to the theory we are discussing in this section. In the BPS limit, the solutions [43] suffer from a naked power-law curvature singularity and hence are not black holes. They are often referred as “superstars.” We now consider these superstars, whose generalization to arbitrary dimensions were given in section 3.3. If we turn on only the $F_{(2)}$ charge, the solution of the $f(R)$ theory is given by

$$\begin{aligned} ds_5^2 &= -H^{-1} h dt^2 + H \left(\frac{dr^2}{h} + r^2 d\Omega_3^2 \right), & F_{(2)} &= \sqrt{2} dt \wedge dH^{-1}, \\ h &= 1 + g^2 r^2 H^2, & F &= \frac{|r|}{\sqrt{r^2 + q^2}}, & H &= 1 + \frac{q}{r^2}. \end{aligned} \quad (124)$$

Since we have

$$H^{-1} h = \frac{r^2 + g^2 (r^2 + q^2)^2}{r^2 + q^2}, \quad (125)$$

it follows that the solution describes a wormhole with r runs from $-\infty$ to $+\infty$. The positivity of F requires that a delta-function matter source at $r = 0$ is needed for supporting this wormhole. Compared to the previous domain wall solution, this charged solution is more accurately called a wormhole since the $r = \text{constant}$ slice is $S^3 \times \mathbb{R}^t$ rather than the four-dimensional Minkowski spacetime. (Note that if we turn on both $F_{(2)}$ and $\mathcal{F}_{(2)}$, the metric around $r = 0$ becomes $ds^2 \sim r^2 dr^2 + \dots$, implying that r^2 can be negative and hence the solution has a naked singularity when H vanishes.) The corresponding $D = 10$ type IIB solution is given by

$$\begin{aligned} ds_{10}^2 &= \sqrt{\Delta} \left(-\frac{r^2 + g^2(r^2 + q^2)^2}{r^2 + q^2} dt^2 + \frac{(r^2 + q^2) dr^2}{r^2 + g^2(r^2 + q^2)^2} + (r^2 + q^2) d\Omega_3^2 + g^{-2} d\theta^2 \right) \\ &\quad + \frac{1}{g^2 \sqrt{\Delta}} \left(\frac{1}{4} \sin^2 \theta (\sigma_1^2 + \sigma_2^2 + (\sigma_3 + 2A_{(1)})^2) + \frac{r^2}{r^2 + q^2} \cos^2 \theta d\phi^2 \right), \\ \Delta &= \cos^2 \theta + \frac{r^2}{r^2 + q^2} \sin^2 \theta. \end{aligned} \quad (126)$$

It is thus clear that the coordinate r runs from 0 to ∞ in $D = 10$.

There is another solution in the $f(R)$ supergravities that does not exist in the original gauged supergravity. That is the solution with $R = -8g^2$. Since for this case, both f and F vanish, and hence the full equations of motion is reduced to simply a scalar-type equation $R = -8g^2$.

6.2 $D = 4$ gauged supergravity

Four-dimensional maximum gauged supergravity has an $SO(8)$ local gauge group. For the bosonic sector, it is consistent to truncate to the $U(1)^4$ Cartan subsector. (See, *e.g.* [45, 23].) We shall consider the special case where three of the $U(1)$ vectors are set to equal. Following the results given in [23], we find that the Lagrangian is given by

$$e^{-1} \mathcal{L} = R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} e^{-\frac{1}{\sqrt{3}} \phi} F_{(2)}^2 - \frac{1}{4} e^{\sqrt{3} \phi} \mathcal{F}_{(2)}^2 + 3g^2 (e^{-\frac{1}{\sqrt{3}} \phi} + e^{\frac{1}{\sqrt{3}} \phi}). \quad (127)$$

In the $f(R)$ frame, this theory becomes

$$e^{-1} \mathcal{L} = \varphi (R + 3g^2) + \varphi^3 (4g^2 - \frac{1}{4} \mathcal{F}_{(2)}^2) - \frac{1}{4} \varphi^{-1} F_{(2)}^2, \quad (128)$$

Thus we see that aside from the $F_{(3)}$ term, this Lagrangian is the same as the one discussed in section 3.4. Thus the $f(R)$ admits both the worm-brane (29) and charged wormhole (53) solutions with $D = 4$. Using the reduction ansatz given in [23] we can lift the solutions back to $D = 11$. The metric of the smeared M2-brane associated with the worm-brane is given by

$$ds_{11}^2 = \Delta^{2/3} \left(dr^2 + \cosh^4 \left(\frac{1}{2} g r \right) dx^\mu dx_\mu + \frac{4 d\theta^2}{g^2} \right)$$

$$\begin{aligned} & + \frac{4}{g^2 \Delta^{1/3}} \left(\tanh^2(\tfrac{1}{2}gr) \cos^2 \theta d\phi^2 + \sin^2 \theta d\Omega_5^2 \right), \\ \Delta &= \cos^2 \theta + \tanh^2(\tfrac{1}{2}gr) \sin^2 \theta. \end{aligned} \quad (129)$$

The lifting of the charged wormhole solution to $D = 11$ gives rise to

$$\begin{aligned} ds_{11}^2 &= \Delta^{2/3} \left[-\frac{\rho^2 + g^2(\rho^2 + q)^3}{\rho^2 + q^2} dt^2 + (\rho^2 + q)^2 \left(\frac{4d\rho^2}{\rho^2 + g^2(\rho^2 + q)^3} + d\Omega_2^2 \right) \right] \\ &+ \frac{4}{g^2 \Delta^{1/3}} \left[\frac{\rho^2}{\rho^2 + q} \cos^2 \theta d\phi^2 + \sin^2 \theta \left((d\psi + \frac{\sqrt{3}q}{\rho^2 + q^2} dt + 2B_{(1)})^2 + d\Sigma_{\mathbb{CP}^2}^2 \right) \right], \\ \Delta &= \cos^2 \theta + \frac{\rho^2}{\rho^2 + q} \sin^2 \theta. \end{aligned} \quad (130)$$

where $J_{(2)} = dB_{(1)}$ is the Kähler 2-form of the \mathbb{CP}^2 metric $d\Sigma_{\mathbb{CP}^2}^2$. Thus we see that in $D = 11$, both coordinates r and ρ run from 0 to ∞ , whilst in the $D = 4$ $f(R)$ theory, they run from $-\infty$ to $+\infty$.

6.3 Gauged $f(R)$ Kaluza-Klein pseudo-supergravities

It was shown in [13] that the Lagrangian (45) admits Killing spinor equations in general dimensions. Consequently the pseudo-supersymmetric extension of the system was constructed in [15]. In the $f(R)$ frame, we find that the full Lagrangian is given by

$$\begin{aligned} e^{-1} \mathcal{L} &= \varphi \left(R + (D-1)(D-3)g^2 \right) + \varphi^3 \left(-\tfrac{1}{4} \mathcal{F}_{(2)}^2 + g^2(D-1) \right) \\ &+ \varphi s^{ij} \left[\tfrac{1}{2} \bar{\psi}_\mu^i \Gamma^{\mu\nu\rho} \mathcal{D}_\nu(\mathcal{A}) \psi_\rho^j - i \sqrt{\frac{(D-2)\beta}{D-1}} \bar{\lambda}^i \Gamma^{\mu\nu} \mathcal{D}_\mu(\mathcal{A}) \psi_\nu^j + \tfrac{1}{2} \mathcal{D}^\nu(\mathcal{A}) (\bar{\psi}_\mu^i \Gamma^\mu \psi_\nu^j) \right] \\ &+ \varphi^2 t^{ij} \left[\frac{i\sqrt{\beta}}{16} \bar{\psi}_\mu^i \Gamma^{\mu\nu\rho\sigma} \psi_\sigma^j - \frac{i\sqrt{\beta}}{8} \bar{\psi}^{i\nu} \psi^{j\rho} + \sqrt{\frac{D-2}{16(D-1)}} \bar{\psi}_\mu^i \Gamma^{\nu\rho} \Gamma^\mu \lambda^j \right. \\ &\quad \left. + \sqrt{\frac{D-2}{16(D-1)}} \bar{\psi}^{i\nu} \Gamma^\rho \lambda^j - \frac{i\sqrt{\beta}(D-2)}{4(D-1)} \bar{\lambda}^i \Gamma^\nu \Gamma^\rho \lambda^j \right] \mathcal{F}_{\nu\rho} \\ &\varphi u^{ij} \left[-\frac{i\sqrt{\beta}}{8} g \left((D-1)\varphi + (D-2)\varphi^{-1} \right) \bar{\psi}_\mu^i \Gamma^{\mu\nu} \psi_\nu^j \right. \\ &\quad \left. - \tfrac{1}{2} \sqrt{(D-1)(D-2)} g \varphi \bar{\psi}_\mu^i \Gamma^\mu \lambda_j + \tfrac{i}{2} (D-2) \sqrt{\beta} g \varphi \bar{\lambda}^i \lambda^j \right], \end{aligned} \quad (131)$$

where all the fermions are charged under the vector field $\mathcal{A}_{(1)}$, with the covariant derivative on fermions given by

$$\mathcal{D}_\mu(\mathcal{A}) \xi^i = D_\mu \xi^i - \tfrac{1}{4} \beta (D-3) \mathcal{A}_\mu u^{ik} s^{jk} x^j, \quad (132)$$

for any spinor field ξ^i . The pseudo-supersymmetric transformation rules are given by

$$\begin{aligned} \delta \psi_\mu^i &= \mathcal{D}_\mu(\mathcal{A}) \epsilon^i - \frac{i\sqrt{\beta}}{4} t^{ij} s^{kj} \varphi^{-1} \mathcal{F}_{\mu\rho} \Gamma^\rho \epsilon^k + \frac{i\sqrt{\beta}}{2} u^{ij} s^{kj} g \varphi \Gamma_\mu \epsilon^k, \\ \delta \lambda^i &= \frac{i\sqrt{\beta(D-1)}}{2\sqrt{(D-2)}} \left[\varphi^{-1} \Gamma^\mu \partial_\mu \varphi \epsilon^i + \frac{i\sqrt{\beta}}{4} t^{ij} s^{kj} \varphi^{-1} \mathcal{F}_{\mu\nu} \Gamma^{\mu\nu} \epsilon^k \right. \\ &\quad \left. - \frac{i\sqrt{\beta}(D-3)}{2} u^{ij} s^{kj} g (\varphi - \varphi^{-1}) \right], \end{aligned}$$

$$\begin{aligned}
\delta\varphi &= -\frac{i}{4}\sqrt{\frac{(D-2)\beta}{D-1}} s^{ij}\varphi\bar{\lambda}^i\epsilon^j, & \delta e_\mu^a &= \frac{1}{4}s^{ij}\bar{\psi}_\mu^i\Gamma^a\epsilon^j, \\
\delta\mathcal{A}_\mu &= \varphi^{-1}t^{ij}\left[-\frac{i\sqrt{\beta}}{4}\bar{\psi}_\mu^i\epsilon^j + \frac{\beta\sqrt{D-2}}{4\sqrt{D-1}}\bar{\lambda}^i\Gamma_\mu\epsilon^j\right].
\end{aligned} \tag{133}$$

Thus we see that there is no term with any space-time derivative on φ in the full Lagrangian. The variation of φ gives rise to the quadratic solution of φ and hence φ can be solved straightforwardly. Substituting the algebraic equation of φ back to the Lagrangian and pseudo-supersymmetric transformation rules, we obtain the $f(R)$ theory of the gauged Kaluza-Klein pseudo-supergravity. Note that the pseudo-supersymmetric transformation rules on the fermions give rise to the Killing spinor equations (28) discussed earlier.

Note that in this example, the Lagrangian in the $f(R)$ frame is a polynomial of φ with integer powers up to the cubic order in all dimensions. This is because the condition for pseudo-supergravity is much less than supergravities for which examples with integer powers are rare and dimensional dependent.

To be self-contained, it is necessary that we present the Γ -matrix and fermion conventions. We adopt exactly the same convention given in [14, 15], which follows the convention of [47]. We present the convention in Table 1. In addition to the Γ -matrix symmetries and spinor representations in diverse dimensions, we also present the s^{ij} , t^{ij} and u^{ij} that appear in the construction.

$D \bmod 8$	$C\Gamma^{(0)}$	$C\Gamma^{(1)}$	Spinor	β	s^{ij}	t^{ij}	u^{ij}
0	S	S	M	+1	δ^{ij}	δ^{ij}	ε^{ij}
	S	A	S-M	-1	ε^{ij}	δ^{ij}	ε^{ij}
1	S	S	M	+1	δ^{ij}	δ^{ij}	ε^{ij}
2	S	S	M	+1	δ^{ij}	δ^{ij}	ε^{ij}
	A	S	M	-1	δ^{ij}	ε^{ij}	δ^{ij}
3	A	S	M	-1	δ^{ij}	ε^{ij}	δ^{ij}
4	A	S	M	-1	δ^{ij}	ε^{ij}	δ^{ij}
	A	A	S-M	+1	ε^{ij}	ε^{ij}	δ^{ij}
5	A	A	S-M	+1	ε^{ij}	ε^{ij}	δ^{ij}
6	A	A	S-M	+1	ε^{ij}	ε^{ij}	δ^{ij}
	S	A	S-M	-1	ε^{ij}	δ^{ij}	ε^{ij}
7	S	A	S-M	-1	ε^{ij}	δ^{ij}	ε^{ij}

Table 1: Γ -matrix symmetries and spinor representations in diverse dimensions. The quantities s^{ij}, t^{ij} and u^{ij} that appear in the Lagrangian take either δ^{ij} or ε^{ij} .

The quantities (s, t, u) satisfy the following identities

$$s^{ij}s^{ik} = \delta^{jk}, \quad s^{jk}t^{jl}s^{lm} = t^{km}, \quad s^{jk}t^{jl} = t^{kj}s^{lj}, \quad s^{kj}t^{jl} = t^{kj}s^{jl}. \quad (134)$$

Note that in the above, s and t can interchange, and each can interchange with u and the identities still hold. We also have the following important identities

$$s^{ji}t^{jk}s^{mk}t^{ml} = \beta\delta^{il}, \quad s^{ji}u^{jk}s^{mk}u^{ml} = -\beta\delta^{il}, \quad s^{ji}u^{jk}s^{lk}t^{lm} = \gamma\beta\epsilon^{im}, \quad (135)$$

where

$$\gamma = \begin{cases} +1, & \text{if } t^{ij} = \delta^{ij}, \\ -1, & \text{if } t^{ij} = \epsilon^{ij}. \end{cases} \quad (136)$$

7 A general class of $f(R)$ pseud-supergravities

It was shown [10] that there exist a subclass of $f(R)$ gravities that admit Killing spinor equations

$$D_\mu\epsilon + W\Gamma_\mu\epsilon = 0, \quad (\Gamma^\mu\nabla_\mu F + U)\epsilon = 0, \quad (137)$$

where

$$U = -\frac{(4D(D-1)W^2 + R)f''(R)}{4(D-1)W'}. \quad (138)$$

The function f satisfies the following second-order linear differential equation

$$f'' - \frac{(4(D-1)(D-2)W^2 + R)W'}{(4D(D-1)W^2 + R)W}f' + \frac{W'}{(4D(D-1)W^2 + R)W}f = 0. \quad (139)$$

As in the previous examples [14, 15], we find that the existence of the Killing spinor equations implies that one can perform a pseudo-supersymmetric extension of these $f(R)$ gravities. Introducing pseudo-gravitino and dilatino fields, we find that the full Lagrangian is given by

$$\begin{aligned} e^{-1}\mathcal{L} = & f(R) + F(R)s^{ij} \left[\frac{1}{2}\bar{\psi}_\mu^i\Gamma^{\mu\nu\rho}D_\nu\psi_\rho^j - i\sqrt{\frac{(D-2)\beta}{D-1}}\bar{\lambda}^i\Gamma^{\mu\nu}D_\mu\psi_\nu^j + \frac{1}{2}D^\nu(\bar{\psi}_\mu^i\Gamma^\mu\psi_\nu^j) \right] \\ & + \varphi u^{ij} \left[-\frac{1}{4}\left(2(D-2)W + U\right)\bar{\psi}_\mu^i\Gamma^{\mu\nu}\psi_\nu^j + \frac{i\sqrt{(D-1)(D-2)}}{\sqrt{\beta}}W\bar{\psi}_\mu^i\Gamma^\mu\lambda_j \right. \\ & \left. + \frac{1}{2}\left(-\frac{U'}{f'} + \frac{1}{D-2}U + 2W\right)\bar{\lambda}^i\lambda^j \right]. \end{aligned} \quad (140)$$

The pseudo-supersymmetric transformation rules are given by

$$\begin{aligned} \delta\psi_\mu^i &= \mathcal{D}_\mu\epsilon^i + u^{ij}s^{kj}W\Gamma_\mu\epsilon^k, \\ \delta\lambda^i &= \frac{i\sqrt{\beta(D-1)}}{2\sqrt{(D-2)}}F^{-1}\left[\Gamma^\mu\partial_\mu F\epsilon^i + Uu^{ij}s^{kj}\epsilon^k\right], \end{aligned}$$

$$\delta e_\mu^a = \frac{1}{4} s^{ij} \bar{\psi}_\mu^i \Gamma^a \epsilon^j. \quad (141)$$

It can be shown that the Lagrangian is invariant under the pseudo-supersymmetric transformation rules up to the quadratic order in fermions. Note that in dimensions where $s^{ij} = u^{ij}$, the i and j indices in fermions can be suppressed. The vanishing of the pseudo-supersymmetric variation on ψ and λ gives rise to precisely two Killing spinor equations obtained in [10].

It is worth observing that in the $f(R)$ frame, the dilatino λ is also an auxiliary field. The variation of the Lagrangian (140) with respect to λ gives an algebraic equation for λ . Substituting this λ back to the Lagrangian give an $f(R)$ pseudo-supergravity involving only the metric and gravitino.

8 Conclusions and discussions

In this paper, we have constructed the $f(R)$ formalism of ten-dimensional supergravities by performing a conformal transformation and casting the theories in the $f(R)$ frame. The characteristics of the $f(R)$ frame is that the dilaton scalar becomes auxiliary and hence can be integrated out. In ten dimensions, the $f(R)$ frame coincides with that of M-theory, D2-branes or NS-NS 5-branes. The process of integrating out the auxiliary dilaton to get $f(R)$ gravity is analogous to integrate out the auxiliary tensor fields in the Polyakov string action to obtain the Nambu-Goto action. When the fermionic sector are included, we show up to quadratic order in fermions that the dilaton remains auxiliary. The conclusion holds at the full (quartic) fermionic orders since there is no derivative at this order. Furthermore, the dilatino also becomes auxiliary and can be integrated out. Using the same technique, we also constructed $f(R)$ theories of some $D = 5$ and $D = 4$ gauged supergravities and large classes of pseudo-supergravities. We obtain many examples of BPS p -brane and wormhole solutions in the $f(R)$ theories and analyze their properties.

There are two important issues to address at this stage. The first is whether the $f(R)$ supergravities are equivalent to the corresponding usual supergravities in the Einstein frame. The second is which formalism is more natural. To answer the first question, we note that the main difference of the two formalism is the conformal transformation. If the conformal transformation is non-singular, the two theories are clearly equivalent. However, when the conformal transformation becomes singular, the solution spaces of the two theories become inequivalent. This is because in general relativity, a solution is not only characterized by the local form, but also determined by the global structure. It clear that the local form of

a solution in $f(R)$ supergravity can be transformed in general to that of the corresponding supergravity, the well-defined global structure of the solution in $f(R)$ supergravity can be destroyed in such a transformation if the conformal factor is singular. In this paper, we presented many examples of well-defined solutions in $f(R)$ supergravities that became badly behaved in the Einstein frame. In fact in terms of the number of smooth solutions, the Einstein frame scores the worst.

In ten dimensional supergravities, there is a large collection of p -branes, and many physical properties of various p -brane frames were discussed in [29]. A distinguishing feature we observed about the $f(R)$ frame is that the dilaton in this frame becomes manifestly auxiliary. The $f(R)$ frame coincides with that of M-theory, D2-branes and NS-NS 5-branes. The physical significance of having an auxiliary coupling field for the D2-brane and NS-NS 5-brane theories requires further investigation. It could be a consequence that the theories are intrinsically non-perturbative and hence there is no such a field that gives rise to the perturbative expansion. It is also worth investigating whether this is simply the artefact of lower-energy effective action, or whether it persists when higher-order curvature invariants are included.

To address the second question, we note that while all supergravities with a dilaton can be cast into the $f(R)$ frame, many of them appear to have irrational powers of the scalar field. For example, the bosonic Lagrangian of $\mathcal{N} = 1$, $D = 7$ gauged supergravity in the $f(R)$ frame becomes

$$\begin{aligned} e^{-1}\mathcal{L}_7 = & \varphi R - g^2\varphi^{\frac{7}{5}}\left(\frac{1}{4}\varphi^{-\frac{8\sqrt{6}}{5}} - 2\varphi^{-\frac{3\sqrt{6}}{5}} - 2\varphi^{\frac{2\sqrt{6}}{5}}\right) \\ & - \frac{1}{48}\varphi^{-\frac{1}{5}+\frac{4\sqrt{6}}{5}}F_{(4)}^2 - \frac{1}{4}\varphi^{-\frac{4}{5}+\frac{2\sqrt{6}}{5}}(F_{(2)}^i)^2 + e^{-1}\mathcal{L}_{FFA} \end{aligned} \quad (142)$$

Although the scalar φ is auxiliary in this case, there is no simple way of integrating out this field. In ten dimensions, however, in all supergravities, the scalar φ in the $f(R)$ frame couples to the form fields with integer power. We find that this is because that the $f(R)$ frame is closely related to the Kaluza-Klein S^1 reduction. It turns out that the $f(R)$ frame is the same as the $(D+1)$ -dimensional frame without scaling in the Kaluza-Klein reduction. In other words, the breathing mode in the Kaluza-Klein circle reduction is an auxiliary field and it becomes manifest if we expressed the lower-dimensional theory in the $(D+1)$ -dimensional frame. We demonstrate in the appendix that this is true even when the higher-order Gauss-Bonnet term is considered. However, a generic higher-order curvature term can introduce derivatives on the breathing mode; nevertheless, the lower-dimensional theory still appears to be simplest in the $f(R)$ frame.

Thus from the circle reduction of M-theory to $D = 10$ supergravity, the $f(R)$ frame can be easily understood as the M-theory frame. It can also be easily understood why it also coincides with that of D2-branes and NS-NS 5-branes. From the M-theory point of view, the $f(R)$ frame is clearly a natural frame to work with. However, the question about which frame is more natural to study supergravity can only be addressed when higher-order curvature terms are included in the theory. This is because when higher-order curvature terms are involved, a conformal transformation can dramatically increase the complexity of the action. For example, heterotic supergravity in $D = 10$ is the simplest in the string frame [48]. This suggests that type IIA supergravity with α' correction is best constructed in the string frame. On the other hand, from our discussion on S^1 reduction in the appendix, if we were simply to obtain type IIA supergravity from $D = 11$ with higher-order curvature invariants, it is more natural to perform the reduction in the $f(R)$ frame. This paradox requires further investigation. It is related to the fact that ten-dimensional supergravities play two roles in string and M-theory. One is that they are the low-energy effective theories of string; the other is that they are related to M-theory through the dimensional compactification.

The $f(R)$ theories of supergravities can be viewed as $f(R)$ gravities coupled to matters including the form fields and also fermions. However, in the traditional $f(R)$ theories with matters, the matter fields are typically coupled to the metric, whilst in $f(R)$ supergravities, they are coupled not only to the metric, but also to the curvature. This enlarges the possibility of constructing $f(R)$ theories coupled with matters.

To conclude, we find that there exists the $f(R)$ formalism for supergravities. Our explicit construction of $f(R)$ supergravities allows us to find new BPS solutions and also leads to new questions that are worth further investigation.

Acknowledgement

We are grateful to Chris Pope for useful discussions, and grateful to KITPC, Beijing, for hospitality during the part of this work. Liu is supported in part by the National Science Foundation of China (10875103, 11135006) and National Basic Research Program of China (2010CB833000). Lü is supported in part by the NSFC grant 11175269.

A KK circle reduction without the conformal scaling

Kaluza-Klein circle reduction has been well-studied. It was typically done with the lower-dimensional metric properly scaled by the breathing mode such that the theory in lower dimensions is in the Einstein frame. In this picture, the breathing mode is a dynamical field and cannot be solved algebraically. Here we present the Kaluza-Klein circle reduction from $(D + 1)$ dimensions to D dimensions, with the D -dimensional metric written in the same frame as that in $(D + 1)$ dimensions. The metric ansatz is given by

$$d\hat{s}_{D+1}^2 = ds_D^2 + \varphi^2(dz + \mathcal{A}_{(1)})^2. \quad (143)$$

The lower-dimensional metric, the breathing mode φ and the Kaluza-Klein vector $\mathcal{A}_{(1)}$ are all independent of the coordinate z . The natural choice of the vielbein and their inverse are given by

$$\begin{aligned} \hat{e}^a &= e^a, & e^{\bar{z}} &= \varphi(dz + \mathcal{A}_{(1)}), \\ \hat{E}_a &= \nabla_a - A_a \partial_z, & \hat{E}_{\bar{z}} &= \varphi^{-1} \partial_z, \end{aligned} \quad (144)$$

where the $(D+1)$ -dimensional world and tangent indices are split to $\hat{\mu} = (\mu, z)$ and $\hat{a} = (a, \bar{z})$ respectively. Note that we have $\nabla_a = E_a^\mu \nabla_\mu$ and $\mathcal{A}_a = E_a^\mu \mathcal{A}_\mu$. The components of the spin-connection $\omega^{ab} \equiv \omega_c^{ab} e^c$ are given by

$$\hat{\omega}_{cab} = \omega_{cab}, \quad \omega_{\bar{z}ab} = -\frac{1}{2}\varphi \mathcal{F}_{ab}, \quad \omega_{b\bar{z}a} = \frac{1}{2}\varphi \mathcal{F}_{ab}, \quad \omega_{\bar{z}\bar{z}a} = \varphi^{-1} \nabla_a \varphi. \quad (145)$$

Here we have $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$. Since we are interested in also the reduction of the higher-order curvature terms, we present the reduction on the Riemann tensors as well as the Ricci tensors. The independent non-vanishing components of the Riemann tensor are given by

$$\begin{aligned} \hat{R}^{ab}{}_{cd} &= R^{ab}{}_{cd} - \frac{1}{2}\varphi^2(\mathcal{F}^a{}_b \mathcal{F}_{cd} - \mathcal{F}^a{}_{[c} \mathcal{F}_{d]b}), \\ \hat{R}_{abc\bar{z}} &= \hat{R}_{c\bar{z}ab} = \nabla_{[a}(\varphi \mathcal{F}_{b]c}) - \nabla_c \phi \mathcal{F}_{ab}, \\ \hat{R}_{a\bar{z}b}^{\bar{z}} &= -\varphi^{-1} \nabla_b \nabla_a \varphi + \frac{1}{4}\varphi^2 \mathcal{F}_{cb} \mathcal{F}^c{}_a. \end{aligned} \quad (146)$$

The independent components of the Ricci-tensor are

$$\begin{aligned} \hat{R}_{ab} &= R_{ab} - \varphi^{-1} \nabla_b \nabla_a \varphi - \frac{1}{2}\varphi^2 \mathcal{F}_{cb} \mathcal{F}^c{}_a, \\ \hat{R}_{a\bar{z}} &= -\frac{1}{2}\varphi^{-2} \nabla_c(\varphi^3 \mathcal{F}^c{}_a), \\ \hat{R}_{\bar{z}\bar{z}} &= -\varphi^{-1} \square \varphi + \frac{1}{4}\varphi^2 \mathcal{F}_{(2)}^2. \end{aligned} \quad (147)$$

The Ricci scalar is

$$\hat{R} = R - \frac{1}{4}\varphi^2 \mathcal{F}_{(2)}^2 - 2\varphi^{-1} \square \varphi. \quad (148)$$

Thus we have

$$\mathcal{L}_{D+1} = \sqrt{-\hat{g}} \hat{R} \quad \longrightarrow \quad \mathcal{L}_D = \sqrt{-g} \left(\varphi R - \frac{1}{4} \varphi^3 \mathcal{F}_{(2)}^2 - \square \varphi \right). \quad (149)$$

Note that the last term is a total derivative and hence can be dropped. It is easy to verify that the equations of motion derived from the D -dimensional Lagrangian satisfy the Einstein equations in $(D+1)$ dimensions. In this reduction scheme, the breathing mode φ in D dimensions is an auxiliary field, and can be integrated out. In other words, the D -dimensional theory is in the “ $f(R)$ frame.”

If $(D+1)$ -dimensional theory has higher-order curvature terms, it becomes more apparent that the reduction ansatz (143) is more natural in that it gives rise to the simplest form in the lower-dimensional theory. For a generic higher-order curvature term, the breathing mode can acquire derivatives and cease to be auxiliary. This is analogous to off-shell supergravities where the auxiliary fields acquire dynamics when higher-order supersymmetric invariants are introduced. However, if the higher-order curvature terms have special properties such as being topological, the breathing mode may remain auxiliary. Let us demonstrate this by reducing Einstein gravity with the Gauss-Bonnet term, namely

$$\hat{e}^{-1} \mathcal{L}_{D+1} = \hat{R} - \Lambda_0 + \alpha \hat{E}_{\text{GB}}, \quad (150)$$

where

$$\hat{E}_{\text{GB}} = \hat{R}^2 - 4 \hat{R}^{\hat{\mu}\hat{\nu}} \hat{R}_{\hat{\mu}\hat{\nu}} + \hat{R}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{R}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}. \quad (151)$$

For simplicity and demonstrating the point, let us for now set $\mathcal{A}_{(1)} = 0$. The reduced Gauss-Bonnet term is given by

$$\hat{E}_{\text{GB}} = E_{\text{GB}} + 8\varphi^{-1} \nabla_a \nabla_b \varphi R^{ab} - 4\varphi^{-1} \square \varphi R. \quad (152)$$

It follows that for $\mathcal{A}_{(1)} = 0$, we have

$$\sqrt{-\hat{g}}(\hat{R} - \Lambda_0 + \alpha \hat{E}_{\text{GB}}) = \sqrt{-g} \varphi (R - \Lambda_0 + \alpha E_{\text{GB}}) + \text{total derivative terms}. \quad (153)$$

In other words, there is no term involving a derivative on φ . This is true even when the Kaluza-Klein vector is turned on. The full S^1 reduction of the Lagrangian (150) on S^1 with the reduction (143) is given by

$$\begin{aligned} e^{-1} \mathcal{L}_D = & \varphi (R - \Lambda_0 + \alpha E_{\text{GB}}) - \frac{1}{4} \varphi^3 \mathcal{F}_{(2)}^2 + \alpha \varphi^3 \left(-R^{abcd} (\mathcal{F}_{ab} \mathcal{F}_{cd} - \mathcal{F}_{ac} \mathcal{F}_{db}) \right. \\ & + 2 \nabla_a \mathcal{F}_{bc} \nabla^a \mathcal{F}^{bc} + \frac{1}{3} \square \mathcal{F}_{(2)}^2 - 2 \nabla_a \nabla_b (\mathcal{F}^2)^{ab} + 2 \nabla_b \mathcal{F}^{ba} \nabla^c \mathcal{F}_{ca} \\ & \left. - \frac{10}{3} \nabla_a \mathcal{F}_{bc} \nabla^b \mathcal{F}^{ac} + 4 F^{ab} \nabla_a \nabla^c \mathcal{F}_{cb} - \frac{4}{3} \mathcal{F}^{ab} \nabla^c \nabla_b \mathcal{F}_{ac} \right) \end{aligned}$$

$$+\alpha\varphi^5\left(\frac{1}{4}(\mathcal{F}_{(2)}^2)^2+\frac{1}{2}(\mathcal{F}_{(2)}^2)^{ab}(\mathcal{F}_{(2)}^2)_{ab}\right), \quad (154)$$

where $(\mathcal{F}_{(2)}^2)^{ab} = \mathcal{F}^a{}_c \mathcal{F}^{bc}$. Thus we see that the breathing mode φ is an auxiliary field and can be integrated out.

B A general class of charged black hole solutions

We find that the Lagrangian (48) admits the following charged black hole solution

$$\begin{aligned} ds_D^2 &= -\mathcal{H}^{-\frac{D-3}{D-2}} H^{-\frac{D-1}{D-2}} h dt^2 + \mathcal{H}^{\frac{1}{D-2}} H^{\frac{D-1}{(D-2)(D-3)}} \left(\frac{dr^2}{h} + r^2 d\Omega_{D-2,k}^2 \right), \\ \mathcal{F}_{(2)} &= \sqrt{k} \coth(\sqrt{k} \tilde{\delta}) dt \wedge d\mathcal{H}^{-1}, \quad F_{(2)} = \sqrt{\frac{D-1}{D-3}} \sqrt{k} \coth(\sqrt{k} \delta) dt \wedge dH^{-1}, \\ h &= k - \frac{\mu}{r^{D-3}} + g^2 r^2 \mathcal{H} H^{\frac{D-1}{D-3}}, \quad e^\phi = \left(\frac{\mathcal{H}}{H} \right)^{\sqrt{\frac{D-1}{2(D-2)}}}, \\ \mathcal{H} &= 1 + \frac{\mu \sinh^2(\sqrt{k} \tilde{\delta})}{k r^{D-3}}, \quad H = 1 + \frac{\mu \sinh^2(\sqrt{k} \delta)}{k r^{D-3}}. \end{aligned} \quad (155)$$

Here the parameter k can be 1, 0, or -1 , corresponding to the cases where the foliation in the transverse space have the metric $d\Omega_{D-2,k}^2$ on the unit S^{D-2} , T^{D-2} or H^{D-2} , where H^{D-2} denotes the unit hyperbolic $(D-2)$ -space of constant negative curvature. Note that the solution has a smooth limit for $k=0$ and it remains real when $k=-1$. The horizon of the black hole is located at the largest root of the function h . When h has a double root, the solution becomes extremal. In $D=5$, the solution is a special case of the $U(1)^3$ charged solutions in five-dimensional gauged supergravity [44]. In $D=4$, it is a special case of the $U(1)^4$ charged solutions in four-dimensional gauged supergravity [45]. (See also [46].) Note that when $F_{(2)}=0$, the single-charge Kerr-AdS black holes of (45) was constructed in [49].

There is another limit one can take for $k=1$ and 0 cases. We can send $\mu \rightarrow 0$, $\delta \rightarrow \infty$ and $\tilde{\delta} \rightarrow \infty$, but with $q = \mu \sinh^2 \delta$ and $\tilde{q} = \mu \sinh^2 \tilde{\delta}$ fixed. In the case of $D=4, 5$, this is a BPS limit and the resulting solutions preserve a fraction of the supersymmetry. For $k=0$, this limit implies that the charges are set to zero, and we obtain a domain wall solution. For $k=1$, this limit gives rise to the solution (51). Note that such a limit cannot be taken for the $k=-1$ case. The properties of the solutions were discussed in sections 3.2, 6.1, 6.2. In particular, it was shown that the singular solutions with $q \neq 0$ and $\tilde{q}=0$ becomes smooth charged wormholes in the corresponding $f(R)$ theories.

References

- [1] K.S. Stelle, *Renormalization of higher derivative quantum gravity*, Phys. Rev. **D16**, 953 (1977).
- [2] K.S. Stelle, *Classical gravity with higher derivatives*, Gen. Rel. Grav. **9**, 353 (1978).
- [3] W. Li, W. Song and A. Strominger, *Chiral gravity in three dimensions*, JHEP **0804**, 082 (2008), arXiv:0801.4566 [hep-th].
- [4] E.A. Bergshoeff, O. Hohm and P.K. Townsend, *Massive gravity in three dimensions*, Phys. Rev. Lett. **102**, 201301 (2009), arXiv:0901.1766 [hep-th].
- [5] H. Lü and C.N. Pope, *Critical gravity in four dimensions*, Phys. Rev. Lett. **106**, 181302 (2011) [arXiv:1101.1971 [hep-th]].
- [6] H. Lü, Y. Pang and C.N. Pope, *Conformal gravity and extensions of critical gravity*, Phys. Rev. D **84**, 064001 (2011) [arXiv:1106.4657 [hep-th]].
- [7] S. Nojiri and S.D. Odintsov, *Introduction to modified gravity and gravitational alternative for dark energy*, eConf **C0602061**, 06 (2006) [Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007)] [arXiv:hep-th/0601213].
- [8] T.P. Sotiriou and V. Faraoni, *$f(R)$ theories of gravity*, Rev. Mod. Phys. **82**, 451 (2010) [arXiv:0805.1726 [gr-qc]].
- [9] A.De Felice and S. Tsujikawa, *$f(R)$ theories*, Living Rev. Rel. **13**, 3 (2010) [arXiv:1002.4928 [gr-qc]].
- [10] H. Liu, H. Lü and Z.L. Wang, *$f(R)$ gravities, Killing Spinor equations, “BPS” domain walls and cosmology*, arXiv:1111.6602 [hep-th].
- [11] H. Lü, C.N. Pope and Z.L. Wang, *Pseudo-supersymmetry, consistent sphere reduction and Killing spinors for the bosonic string*, Phys. Lett. B **702**, 442 (2011) [arXiv:1105.6114 [hep-th]].
- [12] H. Lü and Z.L. Wang, *Killing spinors for the bosonic string*, arXiv:1106.1664 [hep-th].
- [13] H. Liu, H. Lü and Z.L. Wang, *Killing spinors for the bosonic string and the Kaluza-Klein theory with scalar potentials*, arXiv:1106.4566 [hep-th].

- [14] H. Lü, C.N. Pope and Z.L. Wang, *Pseudo-supergravity extension of the bosonic string*, Nucl. Phys. B **854**, 293 (2012) [arXiv:1106.5794 [hep-th]].
- [15] H.S. Liu, H. Lü, Z.L. Wang, *Gauged Kaluza-Klein AdS pseudo-supergravity*, Phys. Lett. B **703**, 524 (2011) [arXiv:1107.2659 [hep-th]].
- [16] G.J. Olmo, *Limit to general relativity in $f(R)$ theories of gravity*, Phys. Rev. D **75**, 023511 (2007) [arXiv:gr-qc/0612047].
- [17] L.J. Romans, *Massive $\mathcal{N} = 2a$ supergravity In ten-dimensions*, Phys. Lett. B **169**, 374 (1986).
- [18] D.Z. Freedman and J.H. Schwarz, *$\mathcal{N} = 4$ supergravity theory with local $SU(2) \times SU(2)$ Invariance*, Nucl. Phys. B **137**, 333 (1978).
- [19] A. Salam and E. Sezgin, *Chiral compactification on $minkowski \times S^2$ of $\mathcal{N} = 2$ Einstein-Maxwell supergravity in six-dimensions*, Phys. Lett. B **147**, 47 (1984).
- [20] A.H. Chamseddine and W. A. Sabra, *$D = 7$ $SU(2)$ gauged supergravity from $D = 10$ supergravity*, Phys. Lett. B **476**, 415 (2000) [arXiv:hep-th/9911180].
- [21] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Consistent group and coset reductions of the bosonic string*, Class. Quant. Grav. **20**, 5161 (2003) [hep-th/0306043].
- [22] H. Lü, C.N. Pope, E. Sezgin and K.S. Stelle, *Stainless super p -branes*, Nucl. Phys. B **456**, 669 (1995) [hep-th/9508042].
- [23] M. Cvetič, M.J. Duff, P. Hoxha, J.T. Liu, H. Lü, J.X. Lu, R. Martinez-Acosta, C.N. Pope, H. Sati, T.A. Tuan, *Embedding AdS black holes in ten-dimensions and eleven-dimensions*, Nucl. Phys. B **558**, 96 (1999) [hep-th/9903214].
- [24] L.J. Romans, *The F_4 gauged supergravity in six dimensions*, Nucl. Phys. B **269**, 691 (1986).
- [25] M. Cvetič, S.S. Gubser, H. Lü and C.N. Pope, *Symmetric potentials of gauged supergravities in diverse dimensions and Coulomb branch of gauge theories*, Phys. Rev. D **62**, 086003 (2000) [arXiv:hep-th/9909121].
- [26] Z.W. Chong, H. Lü and C.N. Pope, *BPS geometries and AdS bubbles*, Phys. Lett. B **614**, 96 (2005) [arXiv:hep-th/0412221].

- [27] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, Phys. Lett. B **76**, 409 (1978).
- [28] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, *Ten-dimensional Maxwell-Einstein supergravity, its currents, and the issue of its auxiliary fields*, Nucl. Phys. **B195**, 97 (1982).
- [29] M.J. Duff, R.R. Khuri and J.X. Lu, *String solitons*, Phys. Rept. **259**, 213 (1995) [arXiv:hep-th/9412184].
- [30] H. Lü, C.N. Pope and P.K. Townsend, *Domain walls from anti-de Sitter spacetime*, Phys. Lett. B **391**, 39 (1997) [arXiv:hep-th/9607164].
- [31] D. Youm, *Localized intersecting BPS branes*, arXiv:hep-th/9902208.
- [32] I.C.G. Campbell and P.C. West, $\mathcal{N} = 2$ $D = 10$ *Non-chiral supergravity and its spontaneous compactification*, Nucl. Phys. **B243**, 112 (1984).
- [33] G.W. Gibbons, G.T. Horowitz and P.K. Townsend, *Higher dimensional resolution of dilatonic black hole singularities*, Class. Quant. Grav. **12**, 297 (1995) [arXiv:hep-th/9410073].
- [34] G.W. Gibbons, M.B. Green and M.J. Perry, *Instantons and seven-branes in type IIB superstring theory*, Phys. Lett. B **370**, 37 (1996) [arXiv:hep-th/9511080].
- [35] J.H. Schwarz, *Covariant field equations of chiral $\mathcal{N} = 2$ $D = 10$ supergravity*, Nucl. Phys. B **226**, 269 (1983).
- [36] E. Bergshoeff, C.M. Hull and T. Ortin, *Duality in the type II superstring effective action*, Nucl. Phys. B **451**, 547 (1995) [arXiv:hep-th/9504081].
- [37] H. Lü, C.N. Pope and T.A. Tran, *Five-dimensional $\mathcal{N} = 4$, $SU(2) \times U(1)$ gauged supergravity from type IIB*, Phys. Lett. B **475**, 261 (2000) [arXiv:hep-th/9909203].
- [38] A. Brandhuber and Y. Oz, *The $D4$ - $D8$ brane system and five dimensional fixed points*, Phys. Lett. B **460**, 307 (1999) [arXiv:hep-th/9905148].
- [39] M. Cvetič, H. Lü and C.N. Pope, *Gauged six-dimensional supergravity from massive type IIA*, Phys. Rev. Lett. **83**, 5226 (1999) [arXiv:hep-th/9906221].
- [40] Z.W. Chong, H. Lü and C.N. Pope, *Rotating strings in massive type IIA supergravity*, arXiv:hep-th/0402202.

- [41] A.H. Chamseddine and H. Nicolai, *Coupling the $SO(2)$ supergravity through dimensional reduction*, Phys. Lett. B **96**, 89 (1980).
- [42] M. Günaydin, G. Sierra and P.K. Townsend, *Gauging the $D = 5$ Maxwell-Einstein supergravity theories: more on Jordan algebras*, Nucl. Phys. B **253**, 573 (1985).
- [43] K. Behrndt, A.H. Chamseddine and W.A. Sabra, *BPS black holes in $\mathcal{N} = 2$ five-dimensional AdS supergravity*, Phys. Lett. B **442**, 97 (1998) [hep-th/9807187].
- [44] K. Behrndt, M. Cvetič and W.A. Sabra, *Nonextreme black holes of five-dimensional $\mathcal{N} = 2$ AdS supergravity*, Nucl. Phys. B **553**, 317 (1999) [hep-th/9810227].
- [45] M.J. Duff and J.T. Liu, *Anti-de Sitter black holes in gauged $\mathcal{N} = 8$ supergravity*, Nucl. Phys. B **554**, 237 (1999) [arXiv:hep-th/9901149].
- [46] W.A. Sabra, *Anti-de Sitter BPS black holes in $\mathcal{N} = 2$ gauged supergravity*, Phys. Lett. B **458**, 36 (1999) [arXiv:hep-th/9903143].
- [47] A. Van Proeyen, *Tools for supersymmetry*, hep-th/9910030.
- [48] E.A. Bergshoeff and M. de Roo, *The quartic effective action of the heterotic string and supersymmetry*, Nucl. Phys. **B328**, 439 (1989).
- [49] S.Q. Wu, *General rotating charged Kaluza-Klein AdS black holes in higher dimensions*, Phys. Rev. D **83** (2011) 121502 (R)